

Derived sections of Grothendieck fibrations and the problems of homotopical algebra

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Introduction

The formalism of topological operads [21] appeared as a way to describe the algebraic structure of n -fold loop spaces. A topological operad \mathcal{O} is a symmetric sequence of spaces $\{\mathcal{O}(l)\}_{l \in \mathbb{N}}$, where each $\mathcal{O}(l)$ should be thought as the space of operations with n inputs and one output, supplied with composition maps $\mathcal{O}(l) \times \mathcal{O}(m) \rightarrow \mathcal{O}(l + m - 1)$ satisfying symmetry, associativity and unitality conditions. Of particular interest are little n -disk operads \mathcal{D}_n (defined as little cube operads in [21]), for which $\mathcal{D}_n(l)$ is (homotopy equivalent to) the configuration space of l points in \mathbb{R}^n . Any n -fold loop space X has a structure of an algebra over \mathcal{D}_n , which means there are maps $\mathcal{D}_n(l) \times X^l \rightarrow X$ satisfying certain conditions.

Denote by \mathbf{DVect}_k the category of chain complexes over a field k . Taking singular chains of $\mathcal{D}_n(l)$ produces the \mathbf{DVect}_k -operad \mathbb{E}_n (and indeed, \mathcal{D}_n are often called \mathbb{E}_n -operads in

topological spaces). Algebras over such operads \mathbb{E}_n , that is, chain complexes M together with action maps $\mathbb{E}_n(l) \otimes M^{\otimes l} \rightarrow M$, have been studied with great interest especially in the recent years [18]. An example of an \mathbb{E}_2 -algebra is the cohomological Hochschild complex $CH(A)$ of a DG-algebra A , which appears in many settings, for example in two-dimensional topological conformal field theories [7]. It is important, however, to remark that $CH(A)$ is an \mathbb{E}_2 algebra *up to quasiisomorphism*: the Deligne Conjecture [3, 19, 25] only implies that there exists an operad \mathcal{O} in \mathbf{DVect}_k , quasiisomorphic to \mathbb{E}_2 , which acts on $CH(A)$. The proofs of this result involve, subsequently, a lot of combinatorial work to construct \mathcal{O} , its action on $CH(A)$, and the (chain of) quasiisomorphisms $\mathcal{O} \cong \mathbb{E}_2$.

The bulkiness of the formalism of operads comes from the fact that two quasi-isomorphic or homotopy equivalent operads can be of very different size and complexity yet describe equivalent structures. There is a different approach to \mathbb{E}_n -algebras, and more generally to structures related to configuration spaces, which relies on the machinery of factorisation algebras of [2]. A factorisation algebra \mathcal{F} over a space X consists of, roughly speaking, a presheaf \mathcal{F}_n of complexes of vector spaces on X^n for each power n , together with additional structure. First, there is a map

$$\Delta_*^n \mathcal{F}_n \longrightarrow \mathcal{F}_1 \quad (\text{i})$$

between the restriction $\Delta_*^n \mathcal{F}_n$ of \mathcal{F}_n along the smallest diagonal $\Delta_n : X \rightarrow X^n$, and \mathcal{F}_1 . Second, if we denote by $i_n : U_n \subset X^n$ the complement $\{(x_i) | x_k \neq x_l\}$ to all diagonals, then there are factorisation maps

$$i_n^* \mathcal{F}_n \longrightarrow \mathcal{F}_1 \boxtimes \dots \boxtimes \mathcal{F}_1 \quad (\text{ii})$$

between the restriction of \mathcal{F}_n to U_n and the external product of \mathcal{F}_1 [2], which are required to be quasiisomorphisms. When X is a k -disk, one can prove that \mathbb{E}_n -algebras correspond to those factorisation algebras on X which are moreover constructible (which means that \mathcal{F}_n is locally constant on the strata for the standard stratification of X^n). The notion of factorisation algebra has proved its use and is arguably more natural and canonical than algebras over topological operads. It leads one to ask if there is a general ‘homotopic-algebraic’ formalism which does not suffer from the noncanonicity issues of topological operads and naturally reproduces factorisation algebra approach to a variety of algebraic structures.

For the context of loop spaces, an approach alternative to operads does exist and is very useful in practical applications. In [24], Graeme Segal introduced the notion of a Γ -space. Denote by \mathbf{Fin}_* the category of finite sets and partially defined maps: a map $S \rightarrow T$ in \mathbf{Fin}_* is a map of sets $U \rightarrow T$ defined on a subset $U \subset S$. A Γ -space A is then defined as a functor

$$\mathbf{Fin}_* \xrightarrow{A} \mathbf{Top}$$

to the category of topological spaces \mathbf{Top} , satisfying Segal conditions. To formulate them, fix a one-element set 1 . For a set S and an element $x \in S$, we have the corresponding partial map $\rho_x : S \rightarrow 1$ defined on the subset $\{x\}$. The Segal conditions say that for each $S \in \mathbf{Fin}_*$, the induced map

$$A(S) \xrightarrow{\prod_{x \in S} A(\rho_x)} A(1)^S \quad (\text{iii})$$

is a homotopy equivalence of topological spaces.

For each $S \in \mathbf{Fin}_*$ there is one map $\pi_S : S \rightarrow 1$ defined on the whole of S . We can consider the following span

$$\begin{array}{ccc} & A(S) & \\ \prod A(\rho_x) \swarrow & & \searrow A(\pi_S) \\ A(1)^S & & A(1). \end{array} \quad (\text{iv})$$

The Segal conditions imply that this span represents a morphism in the homotopy category $\mathbf{Ho Top}$. A choice of a homotopy inverse for the left map gives, noncanonically, a multiplication operation $m_S : A(1)^S \rightarrow A(1)$. After projecting to $\mathbf{Ho Top}$ one can check that $A(1)$ becomes a commutative monoid.

However, a Γ -space A carries more information than the homotopy monoid $A(1)$. Segal, just like May with topological operads, used Γ -spaces to describe infinite loop spaces and his delooping machinery. From the modern perspective, a Γ -space is a proper description of a homotopy coherent commutative monoid in topological spaces. In particular, Γ -spaces describe the same structure as \mathbb{E}_∞ [21] algebras in \mathbf{Top} .

Instead of \mathbf{Fin}_* we can consider the opposite of the usual category of simplices Δ . A suitable modification of definitions then permits to model homotopy coherent monoids with no commutativity as certain simplicial spaces $\Delta^{\text{op}} \rightarrow \mathbf{Top}$. Explicit examples include ordinary loop spaces. Moreover, the work of [1] implies that there are categories C such that n -fold loop spaces — examples of \mathcal{D}_n -algebras — can be also modeled as Segal-type objects $C \rightarrow \mathbf{Top}$ for a suitable choice of the category C . In place of \mathbf{Top} , one can consider any homotopical category, that is a category \mathcal{M} with a subcategory of weak equivalences \mathcal{W} , such that \mathcal{M} has (homotopy) products, and define Segal objects as functors to \mathcal{M} with maps (iii) being weak equivalences.

The Segal space approach contrasts with operadic approach in that multiplicative operations $m_S : A(1)^S \rightarrow (1)$ for a Γ -space A are not defined canonically and instead are constructed using the properties of A , while specifying a model \mathcal{O} for \mathbb{E}_∞ -operad in \mathbf{Top} and an algebra over it means supplementing a lot of structure. In particular, for a $|S|$ -element set S , $A(S)$ need not to be equal to $\mathcal{O}(|S|) \times A(1)^S$. The information about multiplication properties in Segal formalism is thus entirely contained in the category \mathbf{Fin}_* . There is much less arbitrary choice left available, and one might hope it would be easier to construct and work with Segal structures rather than with operadic structures. Moreover, there is a great similarity between Segal Γ -spaces and factorisation algebras: for a factorisation algebra \mathcal{F} , the maps (i) and (ii) provide, after passing to stalks, spans just like (iv).

However, if we attempt to extend the formalism of Segal objects to chain complexes, we immediately run into difficulties. To produce maps like (iii) in the Γ -space picture we used the universal property of Cartesian product \times which is not satisfied by the tensor product \otimes_k of \mathbf{DVect}_k .

To deal with this issue, one observes [24, 27] that any symmetric monoidal category \mathcal{M} is a weakly commutative monoid object in \mathbf{Cat} , the category of categories of suitable size. It is then, up to an equivalence, described by a ‘ Γ -category’ $M : \mathbf{Fin}_* \rightarrow \mathbf{Cat}$, with maps (iii) being equivalences of categories. In order not to choose an equivalence between \mathcal{M} and $M(1)$, one has to work either with pseudofunctors from \mathbf{Fin}_* to \mathbf{Cat} , or equivalently, with Grothendieck opfibrations [11, 28] over \mathbf{Fin}_* : either notion encodes a weak \mathbf{Fin}_* -indexed family of categories.

The way [18] to directly produce a Grothendieck opfibration out of a symmetric monoidal category \mathcal{M} with monoidal product \otimes is as follows. Define \mathcal{M}^\otimes to be the category with objects $(S, \{M_x\}_{x \in S})$ where $S \in \mathbf{Fin}_*$ and each M_x is an object of \mathcal{M} . A morphism $(S, \{M_x\}_{x \in S}) \rightarrow (T, \{N_y\}_{y \in T})$ consists of a partially defined map $f : S \rightarrow T$, and for each $y \in T$, of a morphism $\otimes_{x \in f^{-1}(y)} M_x \rightarrow N_y$; when $f^{-1}(y)$ is empty, the monoidal product over it is the unit object. The compositions can then be defined with the help of the coherence isomorphisms for the product $\otimes : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ and the unit object. The natural functor $p : \mathcal{M}^\otimes \rightarrow \mathbf{Fin}_*$ is a Grothendieck opfibration, which means that the assignment $S \mapsto p^{-1}(S) = \mathcal{M}^S$ is functorial in a weak but coherent way.

Given a monoid object $A \in \mathcal{M}$, we may define a section $\mathbf{Fin}_* \rightarrow \mathcal{M}^\otimes$ of $p : \mathcal{M}^\otimes \rightarrow \mathbf{Fin}_*$ as $S \mapsto (S, \{P_x\})$ with each $P_x = A$. Sections of this type can be characterised by putting suitable normalisation conditions on sections of p . However, there is still no evident way to write diagrams for Segal conditions using the language of sections. If we take a map $f : S \rightarrow T$ in \mathbf{Fin}_* , the value of a section A on f is determined by a map $f_! A(S) \rightarrow A(T)$ in \mathcal{M}^T , where $f_! : \mathcal{M}^S \rightarrow \mathcal{M}^T$ is the functor

$$f_! : (S, \{P_x\}_{x \in S}) \mapsto (T, \{Q_y\}_{y \in T}), \quad Q_y = \otimes_{x \in f^{-1}(y)} P_x. \quad (\text{v})$$

In particular, for the map $\pi_S : S \rightarrow 1$ with $\pi_S^{-1}(1) = S$, we directly get multiplication operations $A(1)^{\otimes S} = (\pi_S)_! A(S) \rightarrow A(1)$. To have a Segal description, we would instead like to have an object which produces diagrams like the following:

$$\begin{array}{ccc} & A_{\pi_S} & \\ \swarrow & & \searrow \\ A^{\otimes S} = (\pi_S)_! A(S) & & A(1). \end{array} \quad (\text{vi})$$

The left map may then be required to be a weak equivalence if \mathcal{M} has such.

More generally, in place of $\mathcal{M}^\otimes \rightarrow \mathbf{Fin}_*$ we can consider general Grothendieck opfibrations $\mathcal{E} \rightarrow \mathcal{C}$ and ask if there is a way to define objects which produce, out of a map $f : c \rightarrow c'$ in \mathcal{C} , spans of the form $f_! A(c) \leftarrow A_f \rightarrow A(c')$ (here $f_! : \mathcal{E}(c) \rightarrow \mathcal{E}(c')$ is the transition functor induced by the opfibration property of $\mathcal{E} \rightarrow \mathcal{C}$).

In this paper we define *derived* sections of opfibrations with weak equivalences, which in particular produce diagrams like (vi) above. Let us briefly describe the construction. Recall first that for a category \mathcal{C} , its simplicial replacement [5] \mathbb{C} is defined as the category whose objects are composable sequences $c_0 \rightarrow \dots \rightarrow c_n$ of arrows of \mathcal{C} of arbitrary finite length $n \geq 0$. A morphism between $c_0 \rightarrow \dots \rightarrow c_n$ and $c'_0 \rightarrow \dots \rightarrow c'_m$ consists of an order-preserving map of ordinals $a : [m] \rightarrow [n]$ (here $[i]$ denotes a totally ordered set of $i+1$ elements $0, 1, \dots, i$) such that $c_{a(k)} = c'_k$ for $0 \leq k \leq m$. If, as before, one denotes by Δ the standard category of simplices, then \mathbb{C} is the opposite of the simplex category of the nerve $N\mathcal{C} : \Delta^{\text{op}} \rightarrow \mathbf{Set}$. The assignments $(c_0 \rightarrow \dots \rightarrow c_n) \mapsto c_0$ or $(c_0 \rightarrow \dots \rightarrow c_n) \mapsto c_n$ determine two ‘head’ and ‘tail’ functors $h : \mathbb{C} \rightarrow \mathcal{C}$ and $t : \mathbb{C} \rightarrow \mathcal{C}^{\text{op}}$.

Consider a functor $F : \mathbb{C} \rightarrow \mathcal{M}$ where \mathcal{M} is an arbitrary category with weak equivalences

W. If we take a morphism $f : c \rightarrow c'$ of \mathbb{C} , we then can consider the following span in \mathbb{C} :

$$\begin{array}{ccc} & c \xrightarrow{f} c' & \\ \swarrow & & \searrow \\ c & & c'. \end{array} \quad (\text{vii})$$

Evaluating F on this diagram gives the corresponding span in \mathcal{M} :

$$\begin{array}{ccc} & F(c \xrightarrow{f} c') & \\ \swarrow & & \searrow \\ F(c) & & F(c'). \end{array} \quad (\text{viii})$$

If one requires that $F(c) \longleftarrow F(c \xrightarrow{f} c')$, is an isomorphism, then the span (viii) defines a map from $F(c)$ to $F(c')$, which we denote as $F(f)$. It then makes sense to ask if $F(gf) = F(g)F(f)$ for a composable pair of arrows $c \xrightarrow{f} c' \xrightarrow{g} c''$, or whether $F(id_c) = id_{F(c)}$. Both those conditions will be satisfied if F sends to isomorphisms those maps of \mathbb{C} which have the form

$$(c_0 \rightarrow \dots \rightarrow c_k \rightarrow \dots \rightarrow c_n) \longrightarrow (c_0 \rightarrow \dots \rightarrow c_k)$$

for $0 \leq k \leq n$ (that is, those maps which are determined by the inclusion of $[k]$ as first $k+1$ elements of $[n]$). Such maps of \mathbb{C} are called *anchor maps* in this article. We observe that such a functor F factors uniquely as $\bar{F} \circ h$, where $\bar{F} : \mathbb{C} \rightarrow \mathcal{M}$ is a functor from the original category \mathbb{C} .

If F sends the anchor maps of \mathbb{C} to weak equivalences of \mathcal{M} , spans like (viii) define morphisms in $\text{Ho } \mathcal{M}$, the localisation of \mathcal{M} with respect to its weak equivalences. We may view such a functor $F : \mathbb{C} \rightarrow \mathcal{M}$ as a weakening of the notion of a functor from \mathbb{C} to \mathcal{M} , with spans obtained from objects $c_0 \rightarrow \dots \rightarrow c_n$ of greater length ensuring the coherence of compositions.

Assume now given an opfibration $p : \mathcal{E} \rightarrow \mathbb{C}$. Moreover, we assume that each fibre $\mathcal{E}(c) := p^{-1}(c)$ has weak equivalences, and for each map $f : c \rightarrow c'$, the functor $f_! : \mathcal{E}(c) \rightarrow \mathcal{E}(c')$ induced by the opfibration property, preserves those weak equivalences. Then there exists a functor $p_{\mathbb{C}} : \mathbf{E} \rightarrow \mathbb{C}$, such that $\mathbf{E}(\mathbf{c}_{[n]}) := p_{\mathbb{C}}^{-1}(\mathbf{c}_{[n]}) \cong \mathcal{E}(c_n)$, and that for each $\alpha : \mathbf{c}_{[n]} \rightarrow \mathbf{c}'_{[m]}$, there is a naturally induced functor¹ $\mathbf{E}(\mathbf{c}'_{[m]}) \rightarrow \mathbf{E}(\mathbf{c}_{[n]})$ isomorphic to $t(\alpha)_! : \mathcal{E}(c'_m) \rightarrow \mathcal{E}(c_n)$.

We define a presection of $p : \mathcal{E} \rightarrow \mathbb{C}$ to be a section $B : \mathbb{C} \rightarrow \mathbf{E}$ of the functor $p_{\mathbb{C}}$. Presections form a category $\text{PSect}(\mathbb{C}, \mathcal{E})$ which can be equipped with weak equivalences if p has such. A presection B acting on spans like (vii) produces this diagram in $\mathcal{E}(c')$:

$$\begin{array}{ccc} & B(c \xrightarrow{f} c') & \\ \swarrow & & \searrow \\ f_! B(c) & & B(c'). \end{array}$$

If the left map of this span and all other produced by applying B to the anchor maps of \mathbb{C} , are isomorphisms, then one can prove that B defines an ordinary section $\mathbb{C} \rightarrow \mathcal{E}$ of the original

¹Unlike p , the functor $p_{\mathbb{C}}$ is a Grothendieck *fibration* and describes a contravariant family over \mathbb{C} .

opfibration $p : \mathcal{E} \rightarrow \mathcal{C}$. If B takes anchor maps to weak equivalences, we call such B a derived section. They form a category² $\mathbb{R}\text{Sect}(\mathcal{C}, \mathcal{E}) \subset \text{PSect}(\mathcal{C}, \mathcal{E})$ with induced weak equivalences. This is our category of derived sections. Denote by $\text{Ho } \mathbb{R}\text{Sect}(\mathcal{C}, \mathcal{E})$ its localisation along the weak equivalences.

The standard way to work with localisations like $\text{Ho } \mathbb{R}\text{Sect}(\mathcal{C}, \mathcal{E})$ is to use model categories. However, in examples such as $\mathbf{DVect}_k^\otimes \rightarrow \mathbf{Fin}_*$, while the fibres $\mathbf{DVect}_k^\otimes(S) = \mathbf{DVect}_k^S$ are model categories, the transition functors $(v) f_! : \mathbf{DVect}_k^S \rightarrow \mathbf{DVect}_k^{T_k}$ do not usually preserve limits or colimits: in the basic case when $f : \{x, y\} \rightarrow 1$ is the map sending x and y to the element of 1, $f_! : \mathbf{DVect}_k^{\{x, y\}} = \mathbf{DVect}_k \times \mathbf{DVect}_k \rightarrow \mathbf{DVect}_k$ is the tensor product \otimes , which preserves neither products nor direct sums. This makes impossible applying the existing techniques [12] for opfibrations and putting a model structure on $\text{PSect}(\mathcal{C}, \mathcal{E})$, let alone derived sections. As a consequence, for example, it is difficult to understand the behaviour of $\mathbb{R}\text{Sect}(\mathcal{C}, \mathcal{E})$ under the base change. Namely, given a functor $F : \mathcal{D} \rightarrow \mathcal{C}$, denote by $F^*p : F^*\mathcal{E} \rightarrow \mathcal{D}$ the pullback of the opfibration $p : \mathcal{E} \rightarrow \mathcal{C}$ (F^*p is again an opfibration). There is an induced functor $\mathbb{F}^* : \text{PSect}(\mathcal{C}, \mathcal{E}) \rightarrow \text{PSect}(\mathcal{D}, F^*\mathcal{E})$, it preserves weak equivalences and respects derived sections. For a given F , we may ask if \mathbb{F}^* admits a homotopy adjoint, or is homotopically full and faithful when restricted to derived sections. Unfortunately, answering these questions without any additional structure on p seems impossible.

To deal with this problem, we propose the approach of homotopical Δ -categories, which axiomatise the notion of geometric realisation of simplicial objects. Admittedly, this is a fairly restrictive technical tool specifically adapted to our setting, but nonetheless, examples of Δ -categories include \mathbf{DVect}_k , simplicial vector spaces $\Delta^{\text{op}}\mathbf{Vect}_k$ and some other categories which admit a sufficiently reasonable action of simplicial sets. For \mathbf{DVect}_k and $\Delta^{\text{op}}\mathbf{Vect}_k$, the Δ -category structure interacts nicely with the tensor product, giving rise to what we call homotopical Δ -(op)fibrations. With the use of this technology, we are able to construct, for each $F : \mathcal{D} \rightarrow \mathcal{C}$ and a homotopical Δ -opfibration $\mathcal{E} \rightarrow \mathcal{C}$, a weak equivalence preserving functor $\mathbb{F}_! : \text{PSect}(\mathcal{D}, F^*\mathcal{E}) \rightarrow \text{PSect}(\mathcal{C}, \mathcal{E})$. This functor is not adjoint to \mathbb{F}^* and does not, in general, preserve derived sections. However, $\mathbb{F}_!$ comes together with additional structure, for example a natural transformation $\epsilon : \mathbb{F}_!\mathbb{F}^* \rightarrow id$ on the level of localisation³ $\text{Ho } \mathbb{R}\text{Sect}(\mathcal{C}, \mathcal{E})$ which behaves like the counit of an adjunction data. In this regard, our approach is ideologically similar to the result of Costello [7], who constructs a derived equivalence by providing explicitly two functors together with natural maps which become isomorphisms on the level of localisation. However, Costello's construction is rather ad-hoc; we attempt to be more systematic.

We then restrict our attention to one specific class of examples which is motivated by algebraic geometry [16, 17] and called resolutions in this paper. A functor $F : \mathcal{D} \rightarrow \mathcal{C}$ is a *resolution* if it is an opfibration and for each $c \in \mathcal{C}$ the fibre $\mathcal{D}(c) := F^{-1}(c)$ has contractible nerve. For an example of a resolution, consider a finite CW-complex Y of homotopy type $K(G, 1)$ and denote by BG the fundamental groupoid of Y . Take I to be the partially ordered set associated to a chosen regular cellular decomposition of Y . Choosing a point (say, the centre) of each cell of I and connecting these points by paths when one cell is

²The notation \mathbb{R} comes from the analogy with $\mathbb{R}Hom$ in homological and homotopical algebra.

³Precisely, ϵ is a natural transformation of functors $\mathbb{F}_!\mathbb{F}^*, id : \text{Ho } \mathbb{R}\text{Sect}(\mathcal{C}, \mathcal{E}) \rightarrow \text{Ho } \text{PSect}(\mathcal{C}, \mathcal{E})$.

included in the other defines a functor

$$F : I \rightarrow BG \tag{ix}$$

which is (equivalent to) a resolution. The functor F induces the pullback functor $F^* : \mathcal{D}(BG, k) \rightarrow \mathcal{D}(I, k)$, where $\mathcal{D}(BG, k)$ and $\mathcal{D}(I, k)$ are the derived categories of functors from BG and I to \mathbf{DVect}_k correspondingly (note that $\mathcal{D}(BG, k)$ is the same thing as $Loc(Y, k)$, the derived category of complexes of locally constant sheaves on Y). One can prove that F^* is full and faithful, with its image consisting of those functors $I \rightarrow \mathbf{DVect}_k$ which are 'locally constant', in the sense that they send all morphisms of I to quasiisomorphisms. We also see that $\mathcal{D}(I, k)$ is a good object: it is the category of modules over the finite-dimensional algebra generated by I . In particular, it is simple to construct objects of $\mathcal{D}(I, k)$, which, if locally constant, can provide examples of G -representations.

An example of a functor (ix) arises for the configuration spaces \mathcal{D}_2^n of n points on a 2-disk, which are the classifying spaces for n -braid groups Br_n , and which admit interesting cellular decompositions using the combinatorics of planar trees [15]. This example is of particular importance for Deligne conjecture and factorisation algebras.

For resolutions, the result we prove in this paper is the following. Given a homotopical Δ -opfibration $\mathcal{E} \rightarrow \mathcal{C}$ and a resolution $F : \mathcal{D} \rightarrow \mathcal{C}$, the induced pullback functor for derived sections, $\mathbb{F}^* : \mathbb{R}\mathbf{Sect}(\mathcal{C}, \mathcal{E}) \rightarrow \mathbb{R}\mathbf{Sect}(\mathcal{D}, F^*\mathcal{E})$, is homotopically full and faithful. In some specific cases we may also characterise the essential homotopical image of \mathbb{F}^* . It consists of those derived sections which are locally constant (in the weak sense) when restricted to the simplicial replacements $\mathbb{D}(c)$ of fibres $\mathcal{D}(c)$. In the case when our opfibration $\mathcal{E} \rightarrow \mathcal{C}$ is the constant opfibration $\mathbf{DVect}_k \times \mathcal{C} \rightarrow \mathcal{C}$, we reproduce the result for derived categories discussed above. In general, resolutions serve as a testing case, indicating that $\mathbb{R}\mathbf{Sect}$ is a reasonable thing to consider. For the example of $\mathbf{DVect}_k^\otimes \rightarrow \mathbf{Fin}_*$, it seems that resolutions can be used to provide the proof of Deligne conjecture without operadic considerations; the details are being investigated by the author.

Organisation of the paper. In the first section, we introduce the formalism of homotopical Δ -categories which we need in order to treat the issues arising from the breakdown of model-categorical formalism. The content of this section is not genuinely new and is somehow present in the folklore. For instance, the geometric realisation functor for homotopical categories, in the setting which goes beyond simplicial model categories, has been considered in [4, Appendix].

In the second section, we recall some of the basic notions and constructions related to the theory of Grothendieck (op)fibrations, including the less known constructions of transpose and power (op)fibrations. Since Grothendieck opfibrations are natural tools for encoding the notion of families of categories, we introduce a class of suitably structured opfibrations, called homotopical Δ -opfibrations, which formalise the notion of a covariant family of Δ -categories.

In the third section, we introduce simplicial replacements and then use them to define derived sections. The fourth section deals with the construction of the pushforward functor $\mathbb{F}_!$ and the map $\epsilon : \mathbb{F}_!\mathbb{F}^* \rightarrow id$, the data which one can use to verify if the 'right adjoint' \mathbb{F}^* is full and faithful.

Finally, the fifth section consists of the analysis of the case of a resolution, stating the Theorems 5.4 and 5.5 and outlining their proof. It is proven that in this case, the inverse image on the derived sections is full and faithful on the homotopy level. In addition, under mild assumptions we can characterise the essential image of the inverse image functor \mathbb{F}^* .

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1 Generalities on geometric realisation

Apart from fixing notation, this section sets up the formalism of homotopical Δ -categories. A Δ -category \mathcal{M} is a category together with a comultiplicative action of the simplex category Δ , and for homotopical Δ -categories this action respect weak equivalences both in \mathcal{M} and simplicial sets, and provides us with a convenient version of geometric realisation functor.

Our primary motivation for introducing such formalism is the structure on the category of chain complexes of vector spaces \mathbf{DVect}_k . It is not a simplicial monoidal model category, but nonetheless comes with a Δ -action given by tensor products with the chain complex of n -simplexes and an associated version of geometric realisation functor (see [4] for an alternative discussion of this example).

1.1 Homotopy colimits

Notation 1.1. For any category \mathcal{C} , $x \in \mathcal{C}$ means that x is an object of \mathcal{C} . We also write $f \in \mathcal{C}$ for morphisms $f : x \rightarrow y$ of \mathcal{C} if there is no confusion. The set of morphisms between two objects x, y of \mathcal{C} is denoted $\mathcal{C}(x, y)$. The category of functors $Fun(I, \mathcal{M})$ between two categories I and \mathcal{M} is often denoted as \mathcal{M}^I . Sometimes, given an object $x \in \mathcal{C}$, we denote again by x the functor from the terminal category to \mathcal{C} which picks out x .

From now on, Δ denotes the usual category of simplexes, i.e. the full subcategory of the category of small categories \mathbf{Cat} spanned, for $n \geq 0$, by categories $[n]$ with $n + 1$ objects $0, \dots, n$ and exactly one morphism from i to j whenever $i \leq j$.

By $\mathbf{SSet} = Fun(\Delta^{\text{op}}, \mathbf{Set})$ we denote the category of simplicial sets. We often identify Δ with its image in \mathbf{SSet} by the Yoneda embedding

$$\Delta^\bullet : \Delta \rightarrow \mathbf{SSet} = Fun(\Delta^{\text{op}}, \mathbf{Set}), [n] \mapsto \Delta^n := \Delta^\bullet([n]) = \Delta(-, [n]). \quad (1.1)$$

For a simplicial object $X : \Delta^{\text{op}} \rightarrow \mathcal{M}$ in a category \mathcal{M} , denote $X_n := X([n])$ for any $[n] \in \Delta$, and similarly, for bisimplicial objects $Y : \Delta^{\text{op}} \times \Delta^{\text{op}} \rightarrow \mathcal{M}$, we write $Y_{nm} = Y([n], [m])$. We also write $\Delta^{\text{op}}\mathcal{M} := Fun(\Delta^{\text{op}}, \mathcal{M})$, $(\Delta \times \Delta^{\text{op}})\mathcal{M} := Fun(\Delta \times \Delta^{\text{op}}, \mathcal{M})$ and so on.

Definition 1.2. A *homotopical category* is a pair $(\mathcal{M}, \mathcal{W})$ where \mathcal{M} is a category and \mathcal{W} is a subcategory of \mathcal{M} which contains all objects and isomorphisms. We moreover require that

for a composable pair of morphisms f, g of \mathcal{M} , if any two elements of $\{f, g, gf\}$ are in \mathcal{W} , then the third one is in \mathcal{W} as well.

We call \mathcal{W} the category of *weak equivalences*. A morphism $f : x \rightarrow y$ of \mathcal{M} is a *weak equivalence* if it belongs to \mathcal{W} .

Definition 1.3. For a homotopical category $(\mathcal{M}, \mathcal{W})$ its *localisation* [8, 14] $\mathcal{W}^{-1}\mathcal{M}$ is the category together with a functor $p : \mathcal{M} \rightarrow \mathcal{W}^{-1}\mathcal{M}$ such that any functor $F : \mathcal{M} \rightarrow \mathcal{N}$ which sends maps of \mathcal{W} to isomorphisms of \mathcal{N} , factors through p up to a canonical isomorphism.

We also denote $\mathcal{W}^{-1}\mathcal{M}$ by $\text{Ho } \mathcal{M}$.

The existence of localisation for homotopical categories $(\mathcal{M}, \mathcal{W})$ where \mathcal{M} is not small is a known set-theoretical issue, and one can check that for the examples of interest, in this paper it will not arise.

Definition 1.4. Given two homotopical categories $(\mathcal{M}, \mathcal{W}_\mathcal{M}), (\mathcal{N}, \mathcal{W}_\mathcal{N})$ a functor $F : \mathcal{M} \rightarrow \mathcal{N}$ is *homotopical* iff $F(\mathcal{W}_\mathcal{M}) \subset \mathcal{W}_\mathcal{N}$. Equivalently, F takes weak equivalences of \mathcal{M} to weak equivalences of \mathcal{N} .

Any homotopical functor $F : \mathcal{M} \rightarrow \mathcal{N}$ produces to a functor $\overline{F} : \text{Ho } \mathcal{M} \rightarrow \text{Ho } \mathcal{N}$.

Example 1.5. Some well known examples of homotopical categories are

- the category **SSet** of simplicial sets which can be equipped with a homotopical structure by defining \mathcal{W} to be the subcategory of weak homotopy equivalences [9] of simplicial sets,
- the category **DVect** $_k$ of unbounded chain complexes over a field k , with \mathcal{W} being the subcategory of quasiisomorphisms [14],
- any model category \mathcal{M} with its subcategory of weak equivalences \mathcal{W} .

Definition 1.6. A subcategory $\mathcal{W} \subset \mathcal{M}$ satisfies the *two-out-of-six* property, if given three maps in \mathcal{M} denoted f, g, h , so that they are composable with compositions gf, hg, hgf , if gf, hg are in \mathcal{W} , then all other maps f, g, h and hgf are in \mathcal{W} .

The subcategory of isomorphisms in any category satisfies two-out-of-six. The subcategory of weak equivalences in any model category satisfies two-out-of-six as well [8].

Definition 1.7. For $I \in \mathbf{Cat}$ and a homotopical category $(\mathcal{M}, \mathcal{W})$, the standard homotopical structure $(\mathcal{M}^I, \mathcal{W}_I)$ on the category of functors $A : I \rightarrow \mathcal{M}$ consists of those natural transformations $\alpha : A \rightarrow B$ which are valued in the maps of \mathcal{W} . That is, for each $i \in I$, the map $\alpha(i) : A(i) \rightarrow B(i)$ is a weak equivalence.

1.2 Tensors and Δ -categories

Denote by $\delta : \Delta \rightarrow \Delta \times \Delta$ the diagonal functor for Δ .

Definition 1.8. A Δ -structure on a category \mathcal{M} consists of

1. a functor

$$\otimes : \Delta \times \mathcal{M} \rightarrow \mathcal{M}, \quad ([n], x) \mapsto \Delta^n \otimes x,$$

2. a natural transformation *diag* depicted as a 2-square

$$\begin{array}{ccc} \Delta \times \mathcal{M} & \xrightarrow{\otimes} & \mathcal{M} \\ \delta \times id \downarrow & \Downarrow diag & \uparrow \otimes \\ \Delta \times \Delta \times \mathcal{M} & \xrightarrow{id \times \otimes} & \Delta \times \mathcal{M} \end{array}$$

3. a natural isomorphism of \mathcal{M} -endofunctors: $\Delta^0 \otimes - \xrightarrow{\sim} id_{\mathcal{M}}$.

These data should satisfy the obvious coassociativity and counitality identities. A category \mathcal{M} with a Δ -structure is called a Δ -category if \mathcal{M} is cocomplete and the functor \otimes preserves colimits in the second argument.

Remark 1.9. It is immediate that a Δ -category \mathcal{M} has a **SSet**-enrichment given by the mapping spaces

$$\text{Map}_{\mathcal{M}}(x, y)_n := \mathcal{M}(\Delta^n \otimes x, y).$$

Example 1.10. The terminal category $[0]$ can be equipped with a (trivial) Δ -structure.

Example 1.11. The category \mathbf{DVect}_k is a Δ -category for $\Delta^n \otimes M := C_{\bullet}(\Delta^n) \otimes_k M$, where C_{\bullet} is the chain complex functor. The natural transformation *diag* comes from the Alexander-Whitney map as follows:

$$diag : C_{\bullet}(\Delta^n) \xrightarrow{C_{\bullet}(\delta)} C_{\bullet}(\Delta^n \times \Delta^n) \rightarrow C_{\bullet}(\Delta^n) \otimes_k C_{\bullet}(\Delta^n).$$

Example 1.12. Any simplicial model category \mathcal{M} is a Δ -category in the obvious way.

Proposition 1.13. *If \mathcal{M} is a Δ -category then $\otimes : \Delta \times \mathcal{M} \rightarrow \mathcal{M}$ can be extended uniquely to a functor $\otimes : \mathbf{SSet} \times \mathcal{M} \rightarrow \mathcal{M}$ such that*

1. \otimes preserves colimits in each argument,
2. there is a family of maps

$$a(S, T, x) : (S \times T) \otimes x \rightarrow S \otimes (T \otimes x) \tag{1.2}$$

natural in $S, T \in \mathbf{SSet}$ and $x \in \mathcal{M}$, associative in a suitable sense and so that for each $[n]$, the composition

$$\Delta^n \otimes x \rightarrow (\Delta^n \times \Delta^n) \otimes x \rightarrow \Delta^n \otimes (\Delta^n \otimes x)$$

equals $diag(n, x)$ of Definition 1.8. Moreover, $a(S, T, x)$ is an isomorphism whenever S or T is discrete.

We sometimes call the natural map $a(S, T, x)$ the *action map*.

Proof. Recall that to each simplicial set S we can associate its category of simplexes Δ/S . Its objects are all simplexes of S , represented as maps $\Delta^n \rightarrow S$, and a morphism between two such objects is given by a map $[n] \rightarrow [m]$ in Δ compatible with morphisms to S . Let $s : \Delta/S \rightarrow \Delta$ denote the functor $(\Delta^n \rightarrow S) \mapsto [n]$, and define $S \otimes x := \varinjlim_{\Delta/S} s \otimes x$.

For two $S, T \in \mathbf{SSet}$, we have a canonical map $\Delta/(S \times T) \rightarrow \Delta/S \times \Delta/T$ induced by the two projections $d_s : \Delta/(S \times T) \rightarrow \Delta/S$ and $d_t : \Delta/(S \times T) \rightarrow \Delta/T$. Denote again by $s : \Delta/S \rightarrow \Delta$, $t : \Delta/T \rightarrow \Delta$ and also by $st : \Delta/(S \times T) \rightarrow \Delta$ the corresponding forgetful functors. Then we have a sequence of maps

$$\begin{aligned} (S \times T) \otimes x &\cong \varinjlim_{\Delta/(S \times T)} st \otimes x \rightarrow \varinjlim_{\Delta/S \times T} st \otimes (st \otimes x) \cong \varinjlim_{\Delta/(S \times T)} (s \circ d_s) \otimes ((t \circ d_t) \otimes x) \rightarrow \\ &\rightarrow \varinjlim_{\Delta_S} \varinjlim_{\Delta_T} s \otimes (t \otimes x) \cong S \otimes (T \otimes x). \end{aligned}$$

Given the construction of this morphism, one can witness the naturality and check its associativity; due to the third condition of Definition 1.8, the action map becomes an isomorphism,

$$(S \times T) \otimes x \xrightarrow{\sim} S \otimes (T \otimes x),$$

when any of the two $S, T \in \mathbf{SSet}$ is discrete. This proves the last assertion. \square

Example 1.14. For any cocomplete category \mathcal{M} , there is canonical Δ -structure on $\Delta^{\text{op}}\mathcal{M} = \text{Fun}(\Delta^{\text{op}}, \mathcal{M})$, which produces a strict associative action of simplicial sets. Given a simplicial set K and a simplicial object $X \in \Delta^{\text{op}}\mathcal{M}$, we define

$$(K \otimes X)_n = K_n \otimes X_n = \coprod_{K_n} X_n.$$

Definition 1.15. Given two categories \mathcal{M}, \mathcal{N} with Δ -structures, a Δ -functor $F : \mathcal{M} \rightarrow \mathcal{N}$ is a functor between underlying categories together with a family of morphisms

$$m_F([n], x) : \Delta^n \otimes F(x) \rightarrow F(\Delta^n \otimes x)$$

natural in both $[n]$ and x . It is required to be compatible with the diagonal maps and unit isomorphisms.

Remark 1.16. Equivalently, a Δ -functor $F : \mathcal{M} \rightarrow \mathcal{N}$ is a simplicial functor for the simplicial enrichment mentioned in Remark 1.9. In particular, it is evident that the composition of Δ -functors is naturally a Δ -functor.

Example 1.17. The tensor product of chain complexes

$$\otimes_k : \mathbf{DVect}_k \times \mathbf{DVect}_k \rightarrow \mathbf{DVect}_k,$$

or more generally, for any finite⁴ set S , the S -fold tensor product

$$\otimes_k : \mathbf{DVect}_k^S \rightarrow \mathbf{DVect}_k$$

can be naturally equipped with the structure of a Δ -functor.

⁴ $S = *$ corresponds to the identity functor, $S = \emptyset$ corresponds to the inclusion of k in \mathbf{DVect}_k .

Proposition 1.18. *A Δ -functor on $F : \mathcal{M} \rightarrow \mathcal{N}$ between Δ -categories determines a family of maps $m_F(S, x) : S \otimes F(x) \rightarrow F(S \otimes x)$ natural in $S \in \mathbf{SSet}$ and $x \in \mathcal{M}$, which restricts to $m_F([n], x)$ for $S = \Delta^n$ and respects the action maps of Proposition 1.13.*

Proof. Define $m_F(S, x)$ as

$$S \otimes F(x) \cong \varinjlim_{\Delta/S} s \otimes F(x) \xrightarrow{m_F} \varinjlim_{\Delta/S} F(s \otimes x) \rightarrow F(\varinjlim_{\Delta/S} s \otimes x) \cong F(S \otimes x).$$

Then the result follows. \square

Recall that for a functor $F : I^{\text{op}} \times I \rightarrow \mathcal{M}$, its *coend* [20, IX.6] is defined as the universal object $\int^I F$ in \mathcal{M} together with maps $F(i, i) \rightarrow \int^I F$ for each $i \in I$, such that for any morphism $i \rightarrow i'$, the induced diagram commutes:

$$\begin{array}{ccc} F(i', i) & \longrightarrow & F(i, i) \\ \downarrow & & \downarrow \\ F(i', i') & \longrightarrow & \int^I F. \end{array}$$

Coends exist in \mathcal{M} when \mathcal{M} is cocomplete.

Definition 1.19. Let \mathcal{M} be a Δ -category, and $X : \Delta^{\text{op}} \rightarrow \mathcal{M}$ a simplicial object in \mathcal{M} . Its *geometric realisation* is defined as

$$|X| := \int^{\Delta^{\text{op}}} \Delta^\bullet \otimes X$$

Where Δ^\bullet is the Yoneda functor (1.1). Varying X , we get a functor $|-| : \Delta^{\text{op}}\mathcal{M} \rightarrow \mathcal{M}$.

For $S \in \mathbf{SSet}$ and $A \in \mathcal{M}$, it is evident that the realisation of the simplicial object $[n] \mapsto S_n \otimes A$ is canonically isomorphic to $S \otimes A$.

Proposition 1.20. *For a Δ -functor $f : \mathcal{M} \rightarrow \mathcal{N}$ we have a canonical natural transformation*

$$s_f : |f(-)|_{\mathcal{N}} \rightarrow f| - |_{\mathcal{M}}$$

between the corresponding geometric realisations, where $f : \Delta^{\text{op}}\mathcal{M} \rightarrow \Delta^{\text{op}}\mathcal{N}$ is the induced functor. It is compatible with the composition in the following sense: the pasting of

$$\begin{array}{ccccc} \Delta^{\text{op}}\mathcal{M} & \xrightarrow{f} & \Delta^{\text{op}}\mathcal{N} & \xrightarrow{g} & \Delta^{\text{op}}\mathcal{K} \\ \downarrow & s_f \downarrow & \downarrow & s_g \downarrow & \downarrow \\ \mathcal{M} & \xrightarrow{f} & \mathcal{N} & \xrightarrow{g} & \mathcal{K} \end{array}$$

with vertical functors given by realisations, is equal to s_{gf} .

Proof. A tedious but straightforward check. \square

1.3 Homotopical Δ -categories

For any bisimplicial object $X \in (\Delta^{\text{op}} \times \Delta^{\text{op}})\mathcal{M}$ denote by $\delta^*X \in \Delta^{\text{op}}\mathcal{M}$ the diagonal simplicial object, that is, the pullback of X along the diagonal map $\delta : \Delta^{\text{op}} \rightarrow \Delta^{\text{op}} \times \Delta^{\text{op}}$.

Definition 1.21. A *homotopical Δ -structure* on a category \mathcal{M} consists of

- a homotopical structure given by a subcategory $\mathcal{W} \subset \mathcal{M}$,
- a Δ -structure with the functor $\otimes : \Delta \times \mathcal{M} \rightarrow \mathcal{M}$,

so that the following conditions are satisfied:

1. the subcategory \mathcal{W} satisfies two-out-of-six (Definition 1.6),
2. \mathcal{M} is a Δ -category and \mathcal{W} is preserved by small coproducts,
3. the induced functor $\otimes : \mathbf{SSet} \times \mathcal{M} \rightarrow \mathcal{M}$ respects weak equivalences in each variable,
4. the induced action map (1.2) $a(S, T, x) : (S \times T) \otimes x \rightarrow S \otimes (T \otimes x)$ is a weak equivalence for each $x \in \mathcal{M}$ and $S, T \in \mathbf{SSet}$,
5. the geometric realisation functor $|-| : \Delta^{\text{op}}\mathcal{M} \rightarrow \mathcal{M}$ preserves pointwise weak equivalences and for each bisimplicial object $X \in (\Delta^{\text{op}} \times \Delta^{\text{op}})\mathcal{M}$, the natural composite map

$$\int^{\Delta^{\text{op}}} \Delta^\bullet \otimes \delta^*X \rightarrow \int^{\Delta^{\text{op}}} \Delta^\bullet \otimes (\Delta^\bullet \otimes \delta^*X) \rightarrow \int^{\Delta^{\text{op}} \times \Delta^{\text{op}}} \Delta^\bullet \otimes (\Delta^\bullet \otimes X) \quad (1.3)$$

is a weak equivalence.

A category together with a homotopical Δ -structure is called a *homotopical Δ -category*.

Remark 1.22. The identity (1.3) implies that for any bisimplicial object $B : \Delta^{\text{op}} \times \Delta^{\text{op}} \rightarrow \mathcal{M}$, one may take geometric realisations in any order. See Proposition 1.26 for details.

Example 1.23. Some simplicial model categories \mathcal{M} , for instance simplicial presheaves with injective model structure or simplicial vector spaces, produce examples of homotopical Δ -categories. The nontrivial point here is that the realisation functor $\Delta^{\text{op}}\mathcal{M} \rightarrow \mathcal{M}$ only preserves weak equivalences between Reedy cofibrant objects [9, VII.3.6], but for the model categories just mentioned, all objects of $\Delta^{\text{op}}\mathcal{M}$ are automatically cofibrant.

Example 1.24. The category \mathbf{DVect}_k is a homotopical Δ -category for the Δ -structure of Example 1.11 and \mathcal{W} being the class of quasiisomorphisms. In this case, all simplicial objects are Reedy-cofibrant, and the functor of geometric realisation is known to be left Quillen for the Reedy model structure on simplicial objects [4, Lemma 9.8].

We assemble together some of the properties of geometric realisation. Define the category Δ_∞ as a subcategory of Δ consisting of all objects and maps $f : [m] \rightarrow [n]$ such that $f(m) = n$. One has the adjunction

$$j : \Delta \rightleftarrows \Delta_\infty : i$$

where $j([n]) = [n+1]$ and the adjunction map $id \rightarrow i \circ j$ evaluated on $[n]$ is the inclusion $[n] \hookrightarrow [n+1]$ of $[n]$ as first $n+1$ elements of $[n+1]$. Intuitively, j attaches one more, maximal, element to each $[n]$.

Definition 1.25. A *split-augmented* simplicial object is a functor $\bar{X} : \Delta_{\infty}^{\text{op}} \rightarrow \mathcal{M}$. A simplicial object $X : \Delta^{\text{op}} \rightarrow \mathcal{M}$ admits a (split) augmentation iff $X \cong j^* \bar{X}$ for some $\bar{X} : \Delta_{\infty}^{\text{op}} \rightarrow \mathcal{M}$.

For a *bisimplicial* object $X : \Delta^{\text{op}} \times \Delta^{\text{op}} \rightarrow \mathcal{M}$, we denote by $||X|_2|_1$ its repeated realisation, that is the coend of the functor

$$([i], [j], [k], [l]) \mapsto \Delta^i \otimes (\Delta^j \otimes X_{kl})$$

and by $||X|_1|_2$ its transpose realisation, which is just a repeated realisation of a transposed bisimplicial object $([n], [m]) \mapsto X_{mn}$.

Proposition 1.26. *For a homotopical Δ -category \mathcal{M} , the following is true:*

1. *For any simplicial object X admitting an augmentation \bar{X} , its realisation is weakly equivalent to $X_{-1} := \bar{X}_0$. Precisely, there are weak equivalences*

$$X_{-1} \rightarrow |X| \rightarrow X_{-1}$$

with composition identity that come from the extra maps $X_{-1} \rightarrow X_n$ and $X_n \rightarrow X_{-1}$.

2. *Given a morphism $X \rightarrow Y$ of bisimplicial objects, we have*

$$(|X|_2|_1 \rightarrow |Y|_2|_1) \in \mathcal{W} \Leftrightarrow (|X|_1|_2 \rightarrow |Y|_1|_2) \in \mathcal{W}$$

To distinguish simplicial and bisimplicial objects, we write X_{\bullet} ($[n] \mapsto X_n$) and $X_{\bullet\bullet}$ ($([n], [m]) \mapsto X_{nm}$) for a simplicial and a bisimplicial object, correspondingly.

Proof. The first statement is proven in a few steps. Denote the tensoring of Example 1.14 by $\otimes : \mathbf{SSet} \times \Delta^{\text{op}}\mathcal{M} \rightarrow \Delta^{\text{op}}\mathcal{M}$. We now prove that the structure of a homotopical Δ -category on \mathcal{M} gives rise to a family of weak equivalences $|K \otimes X_{\bullet}| \rightarrow K \otimes |X_{\bullet}|$ natural in $K \in \mathbf{SSet}$ and $X_{\bullet} \in \Delta^{\text{op}}\mathcal{M}$. The maps are constructed as the following sequence:

$$\begin{aligned} K \otimes |X_{\bullet}| &= \left(\int^{\Delta^{\text{op}}} \Delta^{\bullet} \otimes K_{\bullet} \right) \otimes \left(\int^{\Delta^{\text{op}}} \Delta^{\bullet} \otimes X_{\bullet} \right) \cong \int^{\Delta^{\text{op}} \times \Delta^{\text{op}}} \Delta^{\bullet} \otimes (\Delta^{\bullet} \otimes \coprod_{K_{\bullet}} X_{\bullet}) \\ &\leftarrow \int^{\Delta^{\text{op}}} \Delta^{\bullet} \otimes \delta^* \left(\coprod_{K_{\bullet}} X_{\bullet} \right) \cong \int^{\Delta^{\text{op}}} \Delta^{\bullet} \otimes (K \otimes X_{\bullet}) = |K \otimes X_{\bullet}|. \end{aligned}$$

The only non-invertible map in the chain above,

$$\int^{\Delta^{\text{op}} \times \Delta^{\text{op}}} \Delta^{\bullet} \otimes (\Delta^{\bullet} \otimes \coprod_{K_{\bullet}} X_{\bullet}) \leftarrow \int^{\Delta^{\text{op}}} \Delta^{\bullet} \otimes \delta^* \left(\coprod_{K_{\bullet}} X_{\bullet} \right)$$

is a weak equivalence by Definition 1.21.

By definition, a simplicial homotopy equivalence in $\Delta^{\text{op}}\mathcal{M}$ consists of two maps $f : X_{\bullet} \rightarrow Y_{\bullet}$ and $g : Y_{\bullet} \rightarrow X_{\bullet}$, and two diagrams

$$\begin{array}{ccc} X_{\bullet} & & \\ \downarrow & \searrow gf & \\ \Delta^1 \otimes X_{\bullet} & \xrightarrow{h} & X_{\bullet} \\ \uparrow & \nearrow id & \\ X_{\bullet} & & \end{array} \quad \begin{array}{ccc} Y_{\bullet} & & \\ \downarrow & \searrow fg & \\ \Delta^1 \otimes Y_{\bullet} & \xrightarrow{h'} & Y_{\bullet} \\ \uparrow & \nearrow id & \\ Y_{\bullet} & & \end{array}$$

where the vertical maps are induced from the two inclusions $[0] \rightrightarrows [1]$ in Δ . The natural weak equivalence $|K \otimes X_{\bullet}| \rightarrow K \otimes |X_{\bullet}|$ then implies that, after the realisation, the compositions $|g||f|$ and $|f||g|$ are weak equivalences. By two-out-of-six we get that $|g|$ and $|f|$ are weak equivalences as well.

It is known [23, Lemma 4.5.1] that X_{\bullet} admitting an augmentation \bar{X}_{\bullet} in the sense of Definition 1.25 leads to a diagram in $\Delta^{\text{op}}\mathcal{M}$

$$\bar{X}_0 \rightarrow X_{\bullet} \rightarrow \bar{X}_0$$

naturally appearing from the extra morphisms in \bar{X}_{\bullet} . The composition of these maps is the identity id_{X_0} , and both maps can be shown to be simplicial homotopy equivalences in $\Delta^{\text{op}}\mathcal{M}$; they thus become weak equivalences after applying geometric realisation.

For the last statement of the proposition, observe that both maps in question are weakly equivalent (that is, weakly equivalent as objects of the category $\mathcal{M}^{[1]}$ of maps in \mathcal{M}) to the map

$$\int^{\Delta^{\text{op}}} \Delta^{\bullet} \otimes \delta^*(X_{\bullet\bullet}) \rightarrow \int^{\Delta^{\text{op}}} \Delta^{\bullet} \otimes \delta^*(Y_{\bullet\bullet})$$

which finishes the proof. \square

2 Fibrations, opfibrations, sections

The main aim of this section is to provide background material for the language of Grothendieck fibrations (contravariant families of categories) and opfibrations (covariant families). Along with the basic notions, we also explain some operations such as taking powers and transpose fibrations. Some parts of this material are largely folklore with no well-known reference available, so we supply proofs. We finish by considering families of homotopical Δ -categories as described by homotopical Δ -opfibrations.

2.1 Basic notions

Definition 2.1. Let $p : \mathcal{E} \rightarrow \mathcal{C}$ be a functor. A morphism $\alpha : x \rightarrow y$ in \mathcal{E} is *p-Cartesian* [11], or simply *Cartesian*, if, for every morphism $\beta : z \rightarrow y$ of \mathcal{E} such that $p(\beta) = p(\alpha)$, there exists a unique morphism $\gamma : z \rightarrow x$ such that $\beta = \alpha\gamma$ and $p(\gamma) = id_{p(z)}$.

A morphism $\alpha : x \rightarrow y$ in \mathcal{E} is *p-opCartesian* if it is Cartesian for $p^{\text{op}} : \mathcal{E}^{\text{op}} \rightarrow \mathcal{C}^{\text{op}}$.

Definition 2.2. A functor $p : \mathcal{E} \rightarrow \mathcal{C}$ is called a *Grothendieck fibration* [11, 28] (or simply a fibration) of categories iff the following two conditions are satisfied:

- For every morphism $f : a \rightarrow b$ of \mathcal{C} and $y \in \mathcal{E}$ such that $p(y) = b$ there exists a Cartesian morphism $\alpha : x \rightarrow y$ in \mathcal{E} covering α , that is, $p(\alpha) = f$.
- The composition of Cartesian morphisms is a Cartesian morphism.

Dually, p is called an *opfibration* of categories iff $p^{\text{op}} : \mathcal{E}^{\text{op}} \rightarrow \mathcal{C}^{\text{op}}$ is a fibration of categories.

An (op)fibration $q : \mathcal{O} \rightarrow \mathcal{C}$ is *small* if both \mathcal{C} and \mathcal{O} are small.

Construction 2.3. Given a functor E from \mathcal{C} to categories, we produce an opfibration, which we denote $\int E \rightarrow \mathcal{C}$ and call the *Grothendieck construction* [28] of E . An object of $\int E$ is a pair (c, x) of $c \in \mathcal{C}$ and $x \in E(c)$, and a morphism $(c, x) \rightarrow (c', x')$ consists of $f : c \rightarrow c'$ together with a map $\alpha : E(f)(x) \rightarrow x'$ in $E(x')$.

In this paper, we already used the symbol \int for coends; in fact, $\int E$ can be reproduced as a certain coend.

Similar considerations apply in the case of fibrations. For a contravariant category-valued functor F defined on \mathcal{C} , the Grothendieck construction is a fibration $\int F \rightarrow \mathcal{C}$ with same pairs (c, x) serving as objects, but with maps given by $f : c \rightarrow c'$ and $\beta : x \rightarrow F(f)x'$.

Example 2.4. The category Δ/S for a simplicial set S is exactly the domain of the fibration $\int S \rightarrow \Delta$, where we view S as a functor $\Delta^{\text{op}} \rightarrow \mathbf{Set} \subset \mathbf{Cat}$ contravariant on Δ .

Denote by \mathbf{Fin}_* the category with objects finite sets, and morphisms $f : S \rightarrow T$ given by partially defined maps. That is, f is a pair (D, \tilde{f}) of a subset $D \subset S$ and a map of sets $\tilde{f} : D \rightarrow T$.

Example 2.5. Consider a strict symmetric monoidal category \mathcal{M} , that is, a category together with an associative commutative unital product \otimes . From this data, we can form a functor M to categories defined on \mathbf{Fin}_* by the assignment $S \mapsto M(S) := \mathcal{M}^S$. For each partially defined map $f : S \rightarrow T$, there is a functor $f_! : \mathcal{M}^S \rightarrow \mathcal{M}^T$, which sends an $\{X_s\}_{s \in S} \in \mathcal{M}^S$ to $\{Y_t\}_{t \in T} \in \mathcal{M}^T$ with $Y_t = \otimes_{s \in f^{-1}(t)} X_s$. When the inverse image is empty, the product is equal to the unit object.

Now we take the Grothendieck construction of M , obtaining $\int M \rightarrow \mathbf{Fin}_*$. The category $\mathcal{M}^{\otimes} := \int M$ is otherwise described as follows. Its objects are $(S, \{X_s\}_{s \in S})$ where $S \in \mathbf{Fin}_*$ and each X_s is an object of \mathcal{M} . A morphism $(S, \{X_s\}_{s \in S}) \rightarrow (T, \{Y_t\}_{t \in T})$ is a partially defined map $f : S \rightarrow T$, and a morphism $\otimes_{s \in f^{-1}(t)} X_s \rightarrow Y_t$ for each $t \in T$.

A general (op)fibration is not equal to $\int E \rightarrow \mathcal{C}$ for some functor E to categories. Nonetheless, consider an opfibration $p : \mathcal{E} \rightarrow \mathcal{C}$. For $c \in \mathcal{C}$, denote by $\mathcal{E}(c) = p^{-1}(c)$ the fibre of p over c . Let $f : c \rightarrow c'$ be a morphism in \mathcal{C} and $x \in \mathcal{E}(c)$. Then we can choose an opCartesian morphism $\alpha : x \rightarrow y$ such that $p(\alpha) = f$. This specifies an object $y \in \mathcal{E}(c')$. By the universal property of opCartesian maps, the assignment $x \mapsto y$ defines a functor $f_! : \mathcal{E}(c) \rightarrow \mathcal{E}(c')$, which is called a transition functor along f . One can check [11, 28] that for each composable pair f, g , there exists a coherence isomorphism $g_! \circ f_! \cong (g \circ f)_!$ such that

for any composable triple of arrows f, g, h , any choice of coherence isomorphisms renders commutative the following diagram:

$$\begin{array}{ccc} h_! g_! f_! & \xrightarrow{\sim} & h_! (gf)_! \\ \sim \downarrow & & \sim \downarrow \\ (hg)_! f_! & \xrightarrow{\sim} & (hgf)_!. \end{array}$$

In the literature, such choice of an assignment $f \mapsto f_!$ together with coherence isomorphisms is called a cleavage.

Example 2.6. Take an arbitrary symmetric monoidal category \mathcal{M} . Define the category \mathcal{M}^\otimes with the same objects and morphisms as in Example 2.5, but now with compositions defined with the help of coherence isomorphisms for \otimes . The forgetful functor $\mathcal{M}^\otimes \rightarrow \mathbf{Fin}_*$ is then an opfibration of categories. It is possible to characterise exactly the opfibrations arising from symmetric monoidal categories using Segal conditions [18, 27].

Definition 2.7. Let $p : \mathcal{E} \rightarrow \mathcal{C}$ and $q : \mathcal{E}' \rightarrow \mathcal{C}$ be two (op)fibrations. A *lax morphism* between p and q is a functor $F : \mathcal{E} \rightarrow \mathcal{E}'$ such that $q \circ F = p$. Such F is called a *Cartesian morphism* if it takes (op)Cartesian morphisms of \mathcal{E} to (op)Cartesian morphisms of \mathcal{E}' . A *section* of an (op)fibration p is a lax morphism from the (op)fibration $id_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ to p .

A morphism between two lax or Cartesian morphisms is a natural transformation $\alpha : F \rightarrow F'$ such that for each x in the domain \mathcal{E} , α_x projects to $id_{p(x)}$.

We denote by $\mathbf{Cart}(\mathcal{E}, \mathcal{E}')$ the category of Cartesian morphisms and by $\mathbf{Sect}(\mathcal{C}, \mathcal{E})$ the category of sections.

Construction 2.8. Take an opfibration $p : \mathcal{E} \rightarrow \mathcal{C}$, and for each $c \in \mathcal{C}$, denote by $c \backslash \mathcal{C}$ the category of objects under c [20]. The forgetful functor $c \backslash \mathcal{C} \rightarrow \mathcal{C}$ is an opfibration. Then the assignment $c \mapsto \mathbf{Cart}(c \backslash \mathcal{C}, \mathcal{E})$ defines a covariant category-valued functor on \mathcal{C} . When \mathcal{C} is small, this construction is inverse up to an equivalence [28] to (Grothendieck) Construction 2.3.

This implies that any opfibration (and, similarly, a fibration) $p : \mathcal{E} \rightarrow \mathcal{C}$ can be, up to an equivalence, replaced by an opfibration $\tilde{p} : \tilde{\mathcal{E}} \rightarrow \mathcal{C}$, for which the assignment $c \mapsto \mathcal{E}(c)$ can be made into a strict functor by a choice of transition functors along maps in \mathcal{C} . We call the opfibrations (similarly, fibrations) with later property *strictly cleavable*.

Definition 2.9. In \mathbf{Fin}_* , a map $\rho : S \rightarrow T$ is *inert* [18] if it is defined on a subset T' of S isomorphic to T , and the restriction $\tilde{\rho} : T' \rightarrow T$ is a bijection.

Example 2.10. Any algebra object A in a symmetric monoidal category \mathcal{M} gives a section $A : \mathbf{Fin}_* \rightarrow \mathcal{M}^\otimes$ by the rule $S \mapsto (A, \dots, A) \in \mathcal{M}^\otimes(S)$. Conversely, consider a section $B : \mathbf{Fin}_* \rightarrow \mathcal{M}^\otimes$ which sends the inert maps to opCartesian maps. Then the value $B(1) \in \mathcal{M}$ of B on a one-element set 1 becomes a commutative monoid. To show this, note that the ‘inert-to-opCartesian’ condition implies that $B(S) \cong (S, B(1)_{s \in S})$ by the means of all

opCartesian maps out of $B(S)$ lying over all inert maps $S \rightarrow 1$. Next, take the map $S \rightarrow 1$ defined on each element of S , and consider its image, $B(S) \rightarrow B(1)$, in \mathcal{M}^\otimes . It factors as

$$B(S) \rightarrow B(1)^{\otimes S} \rightarrow B(1)$$

and gives multiplication maps $B(1)^{\otimes S} \rightarrow B(1)$. Examining the composition of maps in \mathcal{M}^\otimes , it can be checked that everything is determined when S is a two-element set, and associated operation $B(1) \otimes B(1) \rightarrow B(1)$ must be associative, with a unit $I_{\mathcal{M}} \rightarrow B(1)$ given by the value of B on the map $\emptyset \rightarrow 1$ in \mathbf{Fin}_* , and commutative. with commutativity following from the action of B on the nontrivial automorphism of S .

Example 2.11. To explain the term ‘lax morphism’, consider a lax symmetric monoidal functor $F : \mathcal{M} \rightarrow \mathcal{N}$ between two symmetric monoidal categories. It means that there is a natural family of maps $FX \otimes FY \rightarrow F(X \otimes Y)$ and a map $I_{\mathcal{N}} \rightarrow FI_{\mathcal{M}}$ with $I_{\mathcal{N}}, I_{\mathcal{M}}$ unit objects, which satisfy suitable coherence conditions. For opfibrations of Example 2.6 the assignment $(S, \{X_s\}_{s \in S}) \mapsto (S, \{FX_s\}_{s \in S})$ then induces a functor $F^\otimes : \mathcal{M}^\otimes \rightarrow \mathcal{N}^\otimes$ over \mathbf{Fin}_* which does not necessarily preserve Cartesian maps. For example, consider the map $(X, Y) \rightarrow X \otimes Y$, which is opCartesian in \mathcal{M}^\otimes . Its image in \mathcal{N}^\otimes , $(FX, FY) \rightarrow F(X \otimes Y)$, factors through the \mathcal{N}^\otimes -opCartesian map $(FX, FY) \rightarrow FX \otimes FY$ exactly by the lax functor structure of F .

Example 2.12. Let $L : \int \mathcal{E} \rightarrow \int \mathcal{E}'$ be a lax morphism between two Grothendieck constructions of covariant functors $\mathcal{E}, \mathcal{E}' : \mathcal{C} \rightarrow \mathbf{Cat}$. For each $c \in \mathcal{C}$, L specifies a functor $L_c : \mathcal{E}(c) \rightarrow \mathcal{E}'(c)$. For each morphism $f : c \rightarrow c'$, we get a 2-square

$$\begin{array}{ccc} \mathcal{E}(c) & \xrightarrow{L_c} & \mathcal{E}'(c) \\ \mathcal{E}(f) \downarrow & \xleftarrow{L_f} & \downarrow \mathcal{E}'(f) \\ \mathcal{E}(c') & \xrightarrow{L_{c'}} & \mathcal{E}'(c'). \end{array}$$

The natural transformation appears because the image under L of an opCartesian map $X \rightarrow \mathcal{E}(f)X$ ($X \in \mathcal{E}(c)$) may not be opCartesian. Factoring $LX \rightarrow L\mathcal{E}(f)X$,

$$LX \rightarrow \mathcal{E}'(f)LX \rightarrow L\mathcal{E}(f)X,$$

gives $\mathcal{E}'(f)LX \rightarrow L\mathcal{E}(f)X$; for each $X \in \mathcal{E}(c)$, all such maps assemble into L_f . For two composable arrows $f : c \rightarrow c'$, $g : c' \rightarrow c''$, there is a pasting property relating L_f, L_g and L_{gf} similar to the one of Proposition 1.20.

For fibrations, there is a difference on the level of 2-diagrams. Consider $\mathcal{F}, \mathcal{F}' : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$ and take a lax morphism $M : \int \mathcal{F} \rightarrow \int \mathcal{F}'$ of fibrations over \mathcal{C} . For $f : c \rightarrow c'$, we obtain a diagram

$$\begin{array}{ccc} \mathcal{F}(c) & \xrightarrow{M_c} & \mathcal{F}'(c) \\ \mathcal{F}(f) \uparrow & \xrightarrow{M_f} & \uparrow \mathcal{F}'(f) \\ \mathcal{F}(c') & \xrightarrow{M_{c'}} & \mathcal{F}'(c') \end{array}$$

with M_f given by arrows of the form $M\mathcal{F}(f)X \rightarrow \mathcal{F}'(f)MX$.

Definition 2.13. Fix an opfibration $p : \mathcal{E} \rightarrow \mathcal{C}$. Define a category denoted as \mathcal{E}^\top as follows:

1. $Ob(\mathcal{E}^\top) = Ob(\mathcal{E})$
2. A morphism from $x \rightarrow z$ in \mathcal{E}^\top is an isomorphism class of cospans in \mathcal{E}

$$x \longrightarrow y \longleftarrow z$$

such that the left arrow is fiberwise, $p(x \rightarrow y) = id_{p(x)}$, and the right arrow is opCartesian.

There is an evident functor $p^\top : \mathcal{E}^\top \rightarrow \mathcal{C}^{op}$ which sends maps $x \longrightarrow y \longleftarrow z$ to $p(y \longleftarrow z)$. A morphism of \mathcal{E}^\top is p^\top -Cartesian iff it can be represented by a span of the form $y \xrightarrow{id_y} y \longleftarrow z$. The functor p^\top is a fibration, which we call the *transpose fibration* of p .

If $\mathcal{E} \rightarrow \mathcal{C}$ equals $\int E \rightarrow \mathcal{C}$ for a functor $E : \mathcal{C} \rightarrow \mathbf{Cat}$, then $\mathcal{E}^\top \rightarrow \mathcal{C}^{op}$ is equivalent to the (fibrational) Grothendieck construction applied to $E : (\mathcal{C}^{op})^{op} \rightarrow \mathbf{Cat}$ viewed as a contravariant functor on \mathcal{C}^{op} .

Example 2.14. The transpose fibration $(\mathcal{M}^\otimes)^\top \rightarrow \mathbf{Fin}_*^{op}$ of $\mathcal{M}^\otimes \rightarrow \mathbf{Fin}_*$ from Example 2.6 can be constructed by hand just like the original opfibration. Now, to define a lax functor between two such fibrations, $(\mathcal{M}^\otimes)^\top \rightarrow (\mathcal{N}^\otimes)^\top$, we would have to consider *oplax* symmetric monoidal functors between symmetric monoidal categories. Such functors preserve (cocommutative) coalgebra objects, as opposed to lax symmetric monoidal functors, which preserve commutative algebra objects. In particular, any coalgebra object C of \mathcal{M} produces a section of $(\mathcal{M}^\otimes)^\top \rightarrow \mathbf{Fin}_*^{op}$ by the assignment $S \mapsto (S, \{C\}_S)$.

Given a functor $F : \mathcal{D} \rightarrow \mathcal{C}$, we can pull back (op)fibrations over \mathcal{C} to \mathcal{D} , with the result again being (op)fibrations. Similarly, given a section $A : \mathcal{C} \rightarrow \mathcal{E}$ of an (op)fibration $\mathcal{E} \rightarrow \mathcal{C}$ we obtain from it the section $F^*A : \mathcal{D} \rightarrow F^*\mathcal{E}$ of the pullback (op)fibration $F^*\mathcal{E} \rightarrow \mathcal{D}$. This operation defines the pullback functor $F^* : \mathbf{Sect}(\mathcal{C}, \mathcal{E}) \rightarrow \mathbf{Sect}(\mathcal{D}, F^*\mathcal{E})$.

Proposition 2.15. Assume given a fibration $\mathcal{F} \rightarrow \mathcal{C}$ and a natural transformation $\alpha : F \rightarrow G$ of functors $F, G : \mathcal{D} \rightarrow \mathcal{C}$. Then

- there is a natural Cartesian map of fibrations $R_\alpha : G^*\mathcal{F} \rightarrow F^*\mathcal{F}$, which we call the restriction map,
- given a section $A : \mathcal{C} \rightarrow \mathcal{F}$, there is a natural morphism of sections

$$F^*A \rightarrow R_\alpha G^*A.$$

The fact that $\mathcal{F} \rightarrow \mathcal{C}$ is a fibration, and not an opfibration, is important for the direction of the arrows in the proposition.

Proof. Up to an equivalence we can assume $\mathcal{F} \rightarrow \mathcal{C}$ to be strictly cleavable. Take $d \in \mathcal{D}$. For each object X of $\mathcal{E}(G(d)) = G^*\mathcal{F}(d)$, we have a Cartesian arrow $Y \rightarrow X$ in \mathcal{F} over

$\alpha_d : F(d) \rightarrow G(d)$. The value $R_\alpha X$ is then defined to be equal to Y ; its action on morphisms can be defined similarly.

Given a section A , its value on $\alpha_d : F(d) \rightarrow G(d)$ can be naturally factored as

$$F^*A(d) = A(F(d)) \rightarrow R_\alpha A(G(d)) \rightarrow A(G(d)) = G^*A(d).$$

Varying d , the arrows $F^*A(d) \rightarrow R_\alpha A(G(d)) = (R_\alpha G^*A)(d)$ define the natural transformation in question. \square

Definition 2.16. Let $p : \mathcal{E} \rightarrow \mathcal{C}$ be an (op)fibration and $I \in \mathbf{Cat}$ a category.

- A product of I and $p : \mathcal{E} \rightarrow \mathcal{C}$ is the functor $I \times p : I \times \mathcal{E} \rightarrow \mathcal{C}$, $(i, x) \mapsto p(x)$.
- A powering of p with I is the functor $p^I : \mathcal{E}^I \rightarrow \mathcal{C}$ where \mathcal{E}^I is the subcategory of $\mathbf{Fun}(I, \mathcal{E})$ consisting of all functors $F : I \rightarrow \mathcal{E}$ such that $p \circ F$ is a constant functor $I \rightarrow \mathcal{C}$.

Both these functors are (op)fibrations.

Unfortunately, the choice of notation such as \mathcal{E}^I may lead to confusion as before we used it to denote the whole category of functors $I \rightarrow \mathcal{E}$. We thus adopt a convention that for (op)fibrations, the powering notation works in the sense of definition above and not otherwise⁵.

Definition 2.17. Given a fibration $p : \mathcal{F} \rightarrow \mathcal{C}$ and an opfibration $q : \mathcal{O} \rightarrow \mathcal{C}$ with small fibres, a *power fibration* $p^q : \mathcal{F}^\mathcal{O} \rightarrow \mathcal{C}$ is defined as follows. An object of $\mathcal{F}^\mathcal{O}$ is a pair of $c \in \mathcal{C}$ and a functor $X : \mathcal{O}(c) \rightarrow \mathcal{F}$ such that pX is constant of value c . A morphism $(c, X) \rightarrow (c', Y)$ consists of $f : c \rightarrow c'$ and a natural transformation $X \rightarrow Y \circ f_!$ of functors $\mathcal{O}(c) \rightarrow \mathcal{F}$ for some choice of transition functor $f_! : \mathcal{O}(c) \rightarrow \mathcal{O}(c')$. The functor $\mathcal{F}^\mathcal{O} \rightarrow \mathcal{C}$ is the natural projection.

One can verify that $\mathcal{F}^\mathcal{O} \rightarrow \mathcal{C}$ is again a fibration, with fibres equivalent to $\mathbf{Fun}(\mathcal{O}(c), \mathcal{F}(c))$. A fibrational transition functor $\mathcal{F}^\mathcal{O}(c') \rightarrow \mathcal{F}^\mathcal{O}(c)$ is given by precomposing an object $F : \mathcal{O}(c) \rightarrow \mathcal{F}(c)$ with $f_! : \mathcal{O}(c) \rightarrow \mathcal{O}(c')$ and postcomposing with $f^* : \mathcal{F}(c') \rightarrow \mathcal{F}(c)$ for some choice of transition functors $f_!$ and f^* in \mathcal{O} and \mathcal{F} respectively.

Lemma 2.18. For a functor $F : \mathcal{D} \rightarrow \mathcal{C}$, and $p : \mathcal{F} \rightarrow \mathcal{C}, q : \mathcal{O} \rightarrow \mathcal{C}$ as above,

1. There is an equivalence of categories

$$\mathbf{Sect}(\mathcal{O}, q^*\mathcal{F}) \cong \mathbf{Sect}(\mathcal{C}, \mathcal{F}^\mathcal{O}).$$

2. There is a Cartesian map

$$(F^*\mathcal{F})^{F^*\mathcal{O}} \rightarrow F^*(\mathcal{F}^\mathcal{O})$$

which is moreover an equivalence over \mathcal{C} .

⁵If we think of ordinary categories as Grothendieck fibrations over a point, then there is no notational ambiguity.

Proof. Clear. □

Proposition 2.19. *Let $p : \mathcal{F} \rightarrow \mathcal{C}$ be a fibration with cocomplete fibres.*

1. *For any functor $X : I^{\text{op}} \times I \rightarrow \mathcal{F}$ such that pX is constant of value $c \in \mathcal{C}$, its coend $\int^I X$ exists in \mathcal{F} and can be calculated in $\mathcal{F}(c)$, defining a lax morphism $\int^I : \mathcal{F}^{I^{\text{op}} \times I} \rightarrow \mathcal{F}$ of fibrations over \mathcal{C} .*
2. *Let*

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{P} & I \times \mathcal{C} \\ & \searrow & \swarrow \\ & \mathcal{C} & \end{array}$$

be an opCartesian morphism of small strictly cleavable opfibrations. Then the obvious functor

$$P^* : \text{Sect}(\mathcal{C}, \mathcal{F}^I) \rightarrow \text{Sect}(\mathcal{C}, \mathcal{F}^0)$$

admits a left adjoint $P_!$.

The fact that $\mathcal{F} \rightarrow \mathcal{C}$ is a fibration is important in this proposition.

Proof. Let $X : I^{\text{op}} \times I \rightarrow \mathcal{F}$ be such that pX is constant at $c \in \mathcal{C}$. Take its coend in $\mathcal{F}(c)$, and denote it $\int_c^I X$. We need to check that it satisfies the universal property of a coend in \mathcal{F} . Let $Z \in \mathcal{F}$ be an object together with maps $X(i, i) \rightarrow Z$ such that for any morphism $i \rightarrow i'$, the diagram below commutes:

$$\begin{array}{ccc} X(i', i) & \longrightarrow & X(i, i) \\ \downarrow & & \downarrow \\ X(i', i') & \longrightarrow & Z. \end{array}$$

Applying $p : \mathcal{F} \rightarrow \mathcal{C}$ to diagrams like above, we find that for each $i \in I$, the map $X(i, i) \rightarrow Z$ lies over a fixed morphism of \mathcal{C} with domain c ; we denote it as $f : c \rightarrow c'$. The fact that p is a fibration then implies that there exists an object f^*Z over c and a Cartesian map $\alpha : f^*Z \rightarrow Z$ covering f , so that each map $X(i, i) \rightarrow Z$ factors through α and the resulting squares

$$\begin{array}{ccc} X(i', i) & \longrightarrow & X(i, i) \\ \downarrow & & \downarrow \\ X(i', i') & \longrightarrow & f^*Z \end{array}$$

are all commutative. This implies the existence of a map $\int_c^I X \rightarrow f^*Z$. One can then see that the composition $\int_c^I X \rightarrow f^*Z \rightarrow Z$ is unique and independent of the choice of $f^*Z \rightarrow Z$ made above. One can see then that this map is the one which ensures the universality of $\int_c^I X$ in the whole of \mathcal{F} .

We now turn to the second statement. We define a functor on the level of fibrations, $P_! : \mathcal{F}^0 \rightarrow \mathcal{F}^I$, by setting $P_!X$ to be the left Kan extension [20] of $X : \mathcal{O}(c) \rightarrow \mathcal{F}(c)$ along the map $P_c : \mathcal{O}(c) \rightarrow I$. Following the line of argumentation as for the coends before, we can see that this Kan extension exists and moreover can be calculated in the fibre $\mathcal{F}(c)$. Given a section $S : \mathcal{C} \rightarrow \mathcal{F}^0$, we apply $P_!$ to its values, inducing the sought-after functor $P_!$, left adjoint to the pullback $P^* : \text{Sect}(\mathcal{C}, \mathcal{F}^I) \rightarrow \text{Sect}(\mathcal{C}, \mathcal{F}^0)$. \square

2.2 Homotopical Δ -opfibrations

Definition 2.20. A *homotopical structure* on an opfibration $\mathcal{E} \rightarrow \mathcal{C}$ consists of a homotopical structure on \mathcal{E} , given by a subcategory $\mathcal{W} \subset \mathcal{E}$ of weak equivalences, compatible with the opfibration in the following sense:

1. the image of \mathcal{W} in \mathcal{C} consists of identity morphisms,
2. in a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ f \downarrow & & \downarrow f' \\ A' & \xrightarrow{\alpha'} & B' \end{array}$$

if we have $f \in \mathcal{W}$ and α, α' are opCartesian, then $f' \in \mathcal{W}$.

The definition of a homotopical fibration is dual, with implication in the diagram above going into opposite direction. One can see that for an (op)fibration with homotopical structure, $\mathcal{W} = \coprod_{c \in \mathcal{C}} \mathcal{W}(c)$, with $(\mathcal{E}(c), \mathcal{W}(c))$ being a homotopical category for each $c \in \mathcal{C}$. The transition functors of the (op)fibration send $\mathcal{W}(c)$ to $\mathcal{W}(c')$.

Definition 2.21. Given an opfibration $\mathcal{E} \rightarrow \mathcal{C}$ with a homotopical structure, a morphism $\alpha : x \rightarrow y$ of \mathcal{E} is *weakly opCartesian* if it can be factored as an opCartesian morphism $x \rightarrow \alpha_!x$ followed by a weak equivalence $\alpha_!x \rightarrow y$. Dually, one has the notion of a weakly Cartesian morphism for homotopical fibrations, where the order of factorisation is reversed.

Before proceeding with the Δ -structure, we need some extra notation. Take an opfibration $\mathcal{E} \rightarrow \mathcal{C}$ and consider its transpose fibration (Definition 2.13) $\mathcal{E}^\top \rightarrow \mathcal{C}^{\text{op}}$. Now, take the product fibration $\Delta \times \mathcal{E}^\top \rightarrow \mathcal{C}^{\text{op}}$. If we have any functor $\otimes : \Delta \times \mathcal{E}^\top \rightarrow \mathcal{E}^\top$ over \mathcal{C}^{op} , then, for a map $f : c \rightarrow c'$ of the original base category \mathcal{C} , we get, after relabelling $\mathcal{E}^\top(c) = \mathcal{E}(c)$ and $\mathcal{E}^\top(f) = f_! : \mathcal{E}(c) \rightarrow \mathcal{E}(c')$, the following diagram

$$\begin{array}{ccc} \Delta \times \mathcal{E}(c) & \xrightarrow{\otimes_c} & \mathcal{E}(c) \\ \text{id} \times f_! \downarrow & \xRightarrow{m_f} & \downarrow f_! \\ \Delta \times \mathcal{E}(c') & \xrightarrow{\otimes_{c'}} & \mathcal{E}(c') \end{array}$$

so that each $m_f : - \otimes_{c'} f_! - \rightarrow f_!(- \otimes_c -)$ appears in the same way as in Example 2.12. Moreover, $f \mapsto m_f$ is suitably functorial in f .

Definition 2.22. A Δ -structure on an *fibration* $\mathcal{F} \rightarrow \mathcal{C}$ is a Δ -structure $\otimes : \Delta \times \mathcal{F} \rightarrow \mathcal{F}$ such that

1. \otimes is a lax morphism of fibrations over \mathcal{C} ,
2. the natural transformation *diag* and unitality isomorphism (see Definition 1.8) are fiberwise.
3. For each $f : c \rightarrow c'$ of \mathcal{C} and a choice of $f^* : \mathcal{F}(c') \rightarrow \mathcal{F}(c)$ via the fibration property, the induced diagram

$$\begin{array}{ccc} \Delta \times \mathcal{F}(c) & \xrightarrow{\otimes_c} & \mathcal{F}(c) \\ \uparrow id \times f^* & \xRightarrow{m_f} & \uparrow f^* \\ \Delta \times \mathcal{F}(c') & \xrightarrow{\otimes_{c'}} & \mathcal{F}(c') \end{array}$$

makes f^* into a Δ -functor.

A Δ -structure on an opfibration $\mathcal{E} \rightarrow \mathcal{C}$ consists of a Δ -structure on its transpose fibration. In particular, the notion is not dual.

Example 2.23. The prototype example for how one ought to think about such Δ -structures is the following. Consider a covariant functor F on \mathcal{C} such that each $F(c)$ is a Δ -category and each $F(c \rightarrow c')$ is a Δ -functor. Forgetting the Δ -structure, we can apply the Grothendieck construction, obtaining an opfibration $\int F \rightarrow \mathcal{C}$. This opfibration is, then, equipped with a Δ -structure inherited from the values of F on the objects and morphisms of \mathcal{C} .

Definition 2.24. A *homotopical* Δ -(op)fibration is an (op)fibration $\mathcal{E} \rightarrow \mathcal{C}$ together with a homotopical structure and a Δ -structure, such that for each $c \in \mathcal{C}$, the induced structure on the fiber $\mathcal{E}(c)$ is that of a homotopical Δ -category.

Example 2.25. Chain complexes give us homotopical Δ -opfibration $\mathbf{DVect}^\otimes \rightarrow \mathbf{Fin}_*$. The Δ -structure on the opfibration is essentially explained in Examples 1.11 and 1.17, and the weak equivalences are simply induced from the quasiisomorphisms of \mathbf{DVect}_k . In the same way, those simplicial model categories which give us a homotopical Δ -structure (Example 1.23) can as well give us homotopical Δ -opfibrations. If such a category \mathcal{M} in addition possesses a compatible monoidal structure (this is true, for example, both for simplicial presheaves and for simplicial vector spaces), then the associated opfibration $\mathcal{M}^\otimes \rightarrow \mathbf{Fin}_*$ is a homotopical Δ -opfibration.

Proposition 2.26. *Let $\mathcal{F} \rightarrow \mathcal{C}$ be a homotopical Δ -fibration. Then*

- *For any functor $F : \mathcal{D} \rightarrow \mathcal{C}$, the pullback $F^*\mathcal{F} \rightarrow \mathcal{D}$ is again a homotopical Δ -fibration.*

- There is a lax realisation morphism of fibrations

$$\begin{array}{ccc}
 \mathcal{F}^{\Delta^{\text{op}}} & \xrightarrow{\quad | - | \quad} & \mathcal{F} \\
 & \searrow \quad \swarrow & \\
 & \mathcal{C} &
 \end{array}$$

such that on each fiber, the functor $\Delta^{\text{op}}\mathcal{F}(c) \rightarrow \mathcal{F}(c)$ is the geometric realisation for the Δ -category $\mathcal{F}(c)$.

Proof. Recall that $\mathcal{F}^{\Delta^{\text{op}}}$ is the subcategory of $\text{Fun}(\Delta^{\text{op}}, \mathcal{F})$ consisting of $X : \Delta^{\text{op}} \rightarrow \mathcal{F}$ which become constant after composing with $\mathcal{F} \rightarrow \mathcal{C}$. Since \mathcal{F} has a Δ -structure, we can consider the assignment $X \mapsto \Delta^{\bullet} \otimes X$, which defines a lax morphism $\mathcal{F}^{\Delta^{\text{op}}} \rightarrow \mathcal{F}^{\Delta \times \Delta^{\text{op}}}$ of fibrations over \mathcal{C} (it can be seen to not preserve Cartesian arrows). We then use the first part of Proposition 2.19 and compose the just-obtained functor with the coend $\mathcal{F}^{\Delta \times \Delta^{\text{op}}} \rightarrow \mathcal{F}$ to obtain the realisation functor of this proposition. \square

Remark 2.27. From the perspective of Δ -structures, we see that it is preferable to consider fibrations and not opfibrations. The motivation for opfibrations as basic ingredients of the play comes from Example 2.10, where algebra objects in a monoidal category \mathcal{M} are presented as sections of $\mathcal{M}^{\otimes} \rightarrow \mathbf{Fin}_*$. Suitably normalised sections of a transpose fibration $(\mathcal{M}^{\otimes})^{\top} \rightarrow \mathbf{Fin}_*^{\text{op}}$ correspond, on the other hand, to coalgebra objects in \mathcal{M} , as can be seen from Example 2.14. The fact that formalism of the subsequent chapter presents derived sections of an opfibration as certain sections of a related fibration may be perceived as an example of certain 'Koszul' duality relating algebraic and coalgebraic objects.

3 Derived sections

This section is devoted to our definition of derived sections. Starting from the introduction of simplicial replacements of categories, we assemble together a few homotopy-theoretic facts to be used later. We also outline how one prolongs an opfibration $\mathcal{E} \rightarrow \mathcal{C}$ to the simplicial replacement \mathbb{C} of \mathcal{C} . We finish by defining presections and derived sections.

As shown in [5] and later in [23], simplicial replacements of categories allow calculating homotopy colimits and may be viewed as certain cofibrant replacements in homotopy-algebraic sense. We will make this perspective on simplicial replacements concrete in our setting through the constructions of the next section.

3.1 Simplicial replacements

Definition 3.1 ([5]). Given a small category \mathcal{C} , its *simplicial replacement*, denoted \mathbb{C} , is the *opposite* of $\Delta/N\mathcal{C} = \int N\mathcal{C}$, that is the opposite of the category of simplexes of the simplicial set $N\mathcal{C}$ (cf. Example 2.4).

An object of \mathbb{C} is a sequence $c_0 \rightarrow \dots \rightarrow c_n$ of composable morphisms in \mathcal{C} . Any functor $F : \mathcal{D} \rightarrow \mathcal{C}$ induces a functor $\mathbb{F} : \mathbb{D} \rightarrow \mathbb{C}$: by the rule $\mathbb{F}(d_0 \rightarrow \dots \rightarrow d_n) = Fd_0 \rightarrow \dots \rightarrow Fd_n$. Observe that \mathbb{F} commutes with the projections from \mathbb{D} and \mathbb{C} to Δ^{op} .

The assignment $\mathbb{C} \mapsto \mathbb{C}$ defines a functor from \mathbf{Cat} to the full subcategory of $\mathbf{Cat}/(\Delta^{\text{op}})$, consisting of opfibrations over Δ^{op} with discrete fibers.

Notation 3.2. We often denote by $\pi : \mathbb{C} \rightarrow \Delta^{\text{op}}$ the natural projection. An object $c_0 \rightarrow \dots \rightarrow c_n$ of \mathbb{C} will be denoted as $\mathbf{c}_{[n]}$ (so that $\pi(\mathbf{c}_{[n]}) = [n]$) or simply as \mathbf{c} when its underlying Δ -object is not important. Given two objects $\mathbf{c}_{[n]}$, $\mathbf{c}'_{[m]}$, and a map $\alpha : c_n \rightarrow c'_0$, we denote by $\mathbf{c}_{[n]} *^\alpha \mathbf{c}'_{[m]}$ the 'concatenated' object

$$c_0 \rightarrow \dots \rightarrow c_n \xrightarrow{\alpha} c'_0 \rightarrow \dots \rightarrow c'_n.$$

Lemma 3.3. *There are functors $h_{\mathbb{C}} : \mathbb{C} \rightarrow \mathcal{C}$ and $t_{\mathbb{C}} : \mathbb{C} \rightarrow \mathcal{C}^{\text{op}}$ given by $\mathbf{c}_{[n]} \mapsto c_0$ or $\mathbf{c}_{[n]} \mapsto c_n$ respectively.* \square

Definition 3.4. A map $\zeta : \mathbf{c}_{[n]} \rightarrow \mathbf{c}'_{[m]}$ is *anchor* iff its projection in Δ , $\pi(\zeta) : [m] \rightarrow [n]$, is an interval inclusion of $[m]$ as first $m+1$ elements of $[n]$, i.e. $\pi(\zeta)(i) = i$ for $0 \leq i \leq m$. In particular, m should be less or equal than n .

A map $\zeta : \mathbf{c}_{[n]} \rightarrow \mathbf{c}'_{[m]}$ is *structural* iff its image under $t_{\mathbb{C}}$ is an identity and the underlying map in Δ^{op} preserves the endpoints: $\pi(\zeta)(m) = n$.

We denote by $A_{\mathbb{C}}$ and $S_{\mathbb{C}}$ the sets of all anchor and structure maps respectively.

Proposition 3.5 (Factorisation and localisation properties). *For a small category \mathcal{C} ,*

1. *Every map $\mathbf{c} \rightarrow \mathbf{c}'$ can be uniquely factored as an anchor map $\mathbf{c} \rightarrow \mathbf{c}''$ followed by a structural map $\mathbf{c}'' \rightarrow \mathbf{c}$.*
2. *Any functor $F : \mathbb{C} \rightarrow \mathcal{N}$ which sends anchor maps of \mathbb{C} to isomorphisms factors essentially uniquely as $F = \tilde{F} \circ h_{\mathbb{C}}$ for $\tilde{F} : \mathcal{C} \rightarrow \mathcal{N}$. In other words, \mathcal{C} is a localisation of \mathbb{C} with respect to anchor maps.*

Proof. The factorisation property is clear and is inherited from Δ . For the second statement, we first note that the functor $h_{\mathbb{C}}^* : \mathcal{N}^{\mathcal{C}} \rightarrow \mathcal{N}^{\mathbb{C}}$ is full and faithful (see e.g. [23, Section 4.4]). This implies that \mathcal{C} is (equivalent to) a localisation of \mathbb{C} with respect to those maps which become isomorphic after applying $h_{\mathbb{C}}$.

Let $F : \mathbb{C} \rightarrow \mathcal{N}$ be a functor which sends $A_{\mathbb{C}}$ to isomorphisms. Define a new functor $\bar{F} : \mathcal{C} \rightarrow \mathcal{N}$. On objects, $\bar{F}(c) = F(c)$ where c is viewed as an object of \mathbb{C} of zero length. Take a span

$$c \longleftarrow (c \xrightarrow{f} c') \longrightarrow c',$$

the action of F on it gives a span $F(c) \leftarrow F(c \rightarrow c') \rightarrow F(c')$. Inverting the left arrow we get a map $\bar{F}(f) : \bar{F}(c) \rightarrow \bar{F}(c')$. The action of F on objects of higher length, $c \rightarrow c' \rightarrow c''$, and on degenerate objects, $c \xrightarrow{id} c$, then ensures that \bar{F} is indeed a functor and $F = \bar{F}h_{\mathbb{C}}$. \square

Remark 3.6. To stress, the class of anchor maps is not saturated in the sense one applies when one speaks of localisation [8]. Indeed, not every map which becomes an isomorphism under $h_{\mathbb{C}}$ is an anchor map.

The proposition permits to justify the idea that, given a homotopical category \mathcal{M} with weak equivalences \mathcal{W} , a functor $F : \mathcal{C} \rightarrow \mathcal{M}$ sending $A_{\mathcal{C}}$ to \mathcal{W} is a suitable weakening of the concept of a functor from \mathcal{C} to \mathcal{M} . The action of F on spans in \mathcal{C} like

$$c \longleftarrow (c \xrightarrow{f} c') \longrightarrow c',$$

where the left arrow is an anchor map, gives a span $F(c) \xleftarrow{\mathcal{W}} F(c \rightarrow c') \rightarrow F(c')$, where the left map is a weak equivalence. On the level of $\mathrm{Ho} \mathcal{M}$, this span gives a map $F(c) \rightarrow F(c')$, which one can denote $F(f)$. Applying F to higher-length objects then ensures higher coherences for the ‘weak functor’ F .

The spans of the form $X \xleftarrow{\mathcal{W}} Y \rightarrow Z$ have appeared before many times in the context of localisation (for example, they are known under the name ‘cocycles’ in [13]). For an arbitrary homotopical category \mathcal{M} , such spans may not constitute a good presentation of morphisms in $\mathrm{Ho} \mathcal{M}$. In practice one may need to make additional assumptions about \mathcal{M} . For our purposes, homotopical Δ -categories provide a sufficient answer.

Given two functors $\mathcal{D} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{B}$, there is a useful categorical notion called the *comma category* F/G [20]. Its objects are triples $(d, b, \alpha : F(d) \rightarrow G(b))$ for $d \in \mathcal{D}$ and $b \in \mathcal{B}$. We need the following adaptation of this notion:

Definition 3.7. Given two a diagram $\mathcal{D} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{B}$, the associated *simplicial comma object* $\mathbb{F} \mathbb{G}$ is defined as the *opposite* of the category $\int F \mathbb{G}$, where $F \mathbb{G} : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \rightarrow \mathbf{Set}$ is the bisimplicial set

$$F \mathbb{G}([n], [m]) = \{\mathbf{d}_{[n]}, \mathbf{b}_{[m]}, \alpha : F(d_n) \rightarrow G(b_0)\}$$

viewed as a contravariant functor to \mathbf{Cat} . In other words, it is the category with objects given by triples $(\mathbf{d}_{[n]}, \mathbf{b}_{[m]}, \alpha : F(d_n) \rightarrow G(b_0))$ and a map

$$(\mathbf{d}_{[n]}, \mathbf{b}_{[m]}, \alpha : F(d_n) \rightarrow G(b_0)) \rightarrow (\mathbf{d}'_{[k]}, \mathbf{b}'_{[l]}, \beta : F(d'_k) \rightarrow G(b'_0))$$

consists of two maps $\mathbf{d}_{[n]} \rightarrow \mathbf{d}'_{[k]}$, $\mathbf{b}_{[m]} \rightarrow \mathbf{b}'_{[l]}$ such that the induced square commutes:

$$\begin{array}{ccc} F(d_n) & \xrightarrow{\alpha} & G(b_0) \\ \uparrow & & \downarrow \\ F(d'_k) & \xrightarrow{\beta} & G(b'_0). \end{array}$$

We often write $\mathbb{D} \mathbb{G}$ or $\mathbb{F} \mathbb{C}$ instead of $\mathbb{F} \mathbb{G}$ if F or G is the identity functor. Given an object $c \in \mathcal{C}$, we also consider $\mathbb{F} \mathbb{C}$ where we treat c as a functor $[0] \rightarrow \mathcal{C}$ and denote its simplicial replacement by the same letter. The canonical functor $\mathbb{F} \mathbb{G} \rightarrow \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ is an opfibration with discrete fibers $\mathbb{F} \mathbb{G}([n], [m]) = F \mathbb{G}([n], [m])$.

There is a *concatenation* functor $con : \Delta \times \Delta \rightarrow \Delta$, $([n], [m]) \mapsto [n] * [m] = [n + m + 1]$, and we think that $[n]$ is included as first $n + 1$ elements of $[n + m + 1]$ and $[m]$ as last $[m + 1]$ elements. The action of con on morphisms is then evident.

Then we observe the following. There is a diagram in **Cat**

$$\begin{array}{ccccc}
 & & \mathbb{F} // \mathbb{G} & & \\
 & \swarrow pr_{\mathbb{D}} & \downarrow \begin{smallmatrix} \Leftarrow \\ \Rightarrow \end{smallmatrix} & \searrow pr_{\mathbb{B}} & \\
 \mathbb{D} & \xrightarrow{\mathbb{F}} & \mathbb{C} & \xleftarrow{\mathbb{G}} & \mathbb{B}
 \end{array} \tag{3.1}$$

with the middle map denoted $pr_{\mathbb{F} // \mathbb{G}}$, covering the diagram

$$\begin{array}{ccccc}
 & & \Delta^{\text{op}} \times \Delta^{\text{op}} & & \\
 & \swarrow \pi_1 & \downarrow \begin{smallmatrix} \Leftarrow \\ \Rightarrow \end{smallmatrix} & \searrow \pi_2 & \\
 \Delta^{\text{op}} & \xrightarrow{id} & \Delta^{\text{op}} & \xleftarrow{id} & \Delta^{\text{op}}
 \end{array} \tag{3.2}$$

with the middle map acting as $([n], [m]) \mapsto [n] * [m]$. Moreover,

- the left natural transformation $pr_{\mathbb{F} // \mathbb{G}} \rightarrow \mathbb{F} \circ pr_{\mathbb{D}}$ is valued in anchor maps $A_{\mathbb{C}}$,
- the right natural transformation $pr_{\mathbb{F} // \mathbb{G}} \rightarrow \mathbb{G} \circ pr_{\mathbb{B}}$ is valued in structural maps $S_{\mathbb{C}}$,
- $pr_{\mathbb{B}}$ is an opfibration whose classifying functor $\mathbb{B} \rightarrow \mathbf{Cat}$ sends anchor maps to equivalences of categories.

All this is evident from Definition 3.7: $pr_{\mathbb{D}}$ maps $(\mathbf{d}_{[n]}, \mathbf{b}_{[m]}, \alpha : F(d_n) \rightarrow G(b_0))$ to $\mathbf{d}_{[n]}$, $pr_{\mathbb{B}}$ maps it to $\mathbf{b}_{[m]}$, and $pr_{\mathbb{F} // \mathbb{G}}$ maps it to $\mathbb{F}(\mathbf{d}_{[n]}) *^{\alpha} \mathbb{G}(\mathbf{b}_{[m]})$.

The fact that $pr_{\mathbb{B}}$ sends anchor maps to equivalences and Proposition 3.5 suggests that that $pr_{\mathbb{B}}$ can be obtained as a pullback of an opfibration over \mathcal{B} along the first element map $h_{\mathcal{B}} : \mathbb{B} \rightarrow \mathcal{B}$. This opfibration $X \rightarrow \mathcal{B}$ consists of the category X whose objects are triples $(\mathbf{d}_{[n]}, b, \alpha : F(d_n) \rightarrow G(b))$ and morphisms are given by $\mathbf{d}_{[n]} \rightarrow \mathbf{d}'_{[m]}$ in \mathbb{D} , $b \rightarrow b'$ in \mathcal{B} , such that the square

$$\begin{array}{ccc}
 F(d_n) & \xrightarrow{\alpha} & G(b) \\
 \uparrow & & \downarrow \\
 F(d'_m) & \xrightarrow{\beta} & G(b')
 \end{array}$$

commutes in \mathcal{B} , and the functor $X \rightarrow \mathcal{B}$ is the projection $(\mathbf{d}_{[n]}, b, \alpha) \mapsto b$.

Let I be a small category and denote by \mathbb{I} its simplicial replacement.

Definition 3.8. Let \mathcal{M} be a homotopical Δ -category. For $X : \mathbb{I} \rightarrow \mathcal{M}$, its *realisation* is defined as $|\Pi X|$, where $|-|$ is the geometric realisation for \mathcal{M} and $\Pi : Fun(\mathbb{I}, \mathcal{M}) \rightarrow \Delta^{\text{op}} \mathcal{M}$ is the left Kan extension along the canonical projection $\pi : \mathbb{I} \rightarrow \Delta^{\text{op}}$.

For any object $i \in I$, there is naturally a map $X(i) \rightarrow |\Pi X|$.

Lemma 3.9. *Let I be a category with a terminal object 1 , and \mathcal{M} be a homotopical Δ -category. Then any $X : \mathbb{I} \rightarrow \mathcal{M}$ sending the anchor maps $A_{\mathbb{I}}$ to maps in \mathcal{W} , the natural map $X(1) \rightarrow |\Pi X|$ is an equivalence.*

Proof. Consider an 'augmented' functor $X^{aug} : \mathbf{i} \mapsto X(\mathbf{i} *^x 1)$ (here x corresponds to the canonical map to the terminal object $t_I(\mathbf{i}) \rightarrow 1$). It is then easy to see that there is a canonical equivalence $X^{aug} \rightarrow X$ coming from the maps $X(\mathbf{i} *^x 1) \rightarrow X(\mathbf{i})$. It then becomes an equivalence of realisations. The object ΠX^{aug} , however, can be completed to a split-augmented simplicial object $\tilde{X}^{aug} : \Delta_{\infty}^{\text{op}} \rightarrow \mathcal{M}$ defined by the formula

$$\tilde{X}_n^{aug} = \Pi X_{n-1}^{aug}, \quad n > 0,$$

$$\tilde{X}_0^{aug} = X(1).$$

in particular, one augmentation map $X(1) \rightarrow \tilde{X}_1^{aug} = \coprod_i X(i \rightarrow 1)$ comes from the image $X(1) \rightarrow X(1 \rightarrow 1)$ of the degeneracy map $1 \rightarrow (1 \rightarrow 1)$ and the other map

$$\tilde{X}_1^{aug} = \coprod_i X(i \rightarrow 1) \rightarrow X(1)$$

is just the coproduct of the natural maps $X(i \rightarrow 1) \rightarrow X(1)$. By Proposition 1.26 we have the equivalences

$$X(1) \rightarrow |\Pi X^{aug}| \rightarrow X(1)$$

and we can see that the composite map $X(1) \rightarrow |\Pi X^{aug}| \rightarrow |\Pi X|$, which is an equivalence, is equal to the map in question. \square

Lemma 3.10. *Let I be a category with contractible nerve and \mathcal{M} be a homotopical Δ -category. If a functor $X : \mathbb{I} \rightarrow \mathcal{M}$ takes all morphisms of \mathbb{I} to isomorphisms, then the natural map $X(i) \rightarrow |\Pi X|$ is an equivalence for any $i \in I$.*

Proof. Fix $i \in I$. Proposition 3.5 implies that the functor X can be factored as $\overline{X} \circ h_I$ with $\overline{X} : I \rightarrow \mathcal{M}$. X moreover factors through the fundamental groupoid of I , which is contractible. One can then see that

$$\Pi X_n = \coprod_{\mathbf{i}_{[n]}} X(\mathbf{i}_{[n]}) \cong \coprod_{\mathbf{i}_{[n]}} (\overline{X} \circ h_I)(\mathbf{i}_{[n]}) \cong \coprod_{\mathbf{i}_{[n]}} X(i_0) \cong \coprod_{\mathbf{i}_{[n]}} X(i),$$

and so $|\Pi X| = NI \otimes X(i)$, which is equivalent to $X(i)$, and the map $X(i) \rightarrow |\Pi X|$ in question is a homotopy inverse of the projection $NI \otimes X(i) \rightarrow X(i)$. \square

Definition 3.11. For an opfibration $\mathcal{E} \rightarrow \mathcal{C}$, its *simplicial extension* is a fibration $\mathbf{E} \rightarrow \mathbb{C}$ which is the pullback of a transpose fibration $\mathcal{E}^{\top} \rightarrow \mathcal{C}^{\text{op}}$ along $t_{\mathcal{C}} : \mathbb{C} \rightarrow \mathcal{C}^{\text{op}}$.

We stress that \mathbf{E} is not a simplicial replacement of \mathcal{E} or \mathcal{E}^{\top} . In particular, the fibre of $\mathbf{E} \rightarrow \mathbb{C}$ over an object $\mathbf{c}_{[n]}$ is equivalent to $\mathcal{E}(c_n)$. If $\mathcal{E} \rightarrow \mathcal{C}$ comes from a functor $\mathcal{E} : \mathcal{C} \rightarrow \mathbf{Cat}$, then $\mathbf{E} \rightarrow \mathbb{C}$ corresponds to the functor

$$\mathbb{C}^{\text{op}} \xrightarrow{t_{\mathcal{C}}^{\text{op}}} \mathcal{C} \xrightarrow{\mathcal{E}} \mathbf{Cat}$$

viewed as a contravariant functor on \mathbb{C} .

Remark 3.12. Given two functors $k_1, k_2 : K \rightarrow \mathbb{C}$ and a natural transformation $\alpha : k_1 \rightarrow k_2$ valued in structural maps $S_{\mathbb{C}}$, we have that the induced Cartesian map of fibrations

$$\alpha^* : k_2^* \mathbf{E} \rightarrow k_1^* \mathbf{E}$$

is in fact an equivalence.

We can also pull back $\mathcal{E} \rightarrow \mathbb{C}$ to \mathbb{C} by the means of the functor $h_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C}$. The link between this pullback and the fibration $\mathbf{E} \rightarrow \mathbb{C}$ is in the following:

Proposition 3.13. *Given an opfibration $p : \mathcal{E} \rightarrow \mathbb{C}$, there is a morphism $T : h_{\mathbb{C}}^* \mathcal{E} \rightarrow \mathbf{E}$ commuting with functors to \mathbb{C} which sends opCartesian maps of $h_{\mathbb{C}}^* \mathcal{E}$ to Cartesian maps of \mathbf{E} and is universal, i.e. any other functor $G : h_{\mathbb{C}}^* \mathcal{E} \rightarrow \mathbf{E}$ over \mathbb{C} with such a property factors through T up to a natural isomorphism.*

Proof. Consider the category \mathcal{X} defined as follows.

- An object of \mathcal{X} is a pair $(\mathbf{c}_{[n]}, \alpha)$ where $\mathbf{c}_{[n]} = c_0 \rightarrow \dots \rightarrow c_n$ is an object of \mathbb{C} and $\alpha : x \rightarrow y$ is an opCartesian map in \mathcal{E} which covers the composition $c_0 \rightarrow c_n$ in \mathbb{C} (i.e. $p(\alpha) = c_0 \rightarrow c_n$),
- A morphism $(\mathbf{c}_{[n]}, \alpha : x \rightarrow y) \rightarrow (\mathbf{c}'_{[m]}, \beta : x' \rightarrow y')$ consists of a map $\mathbf{c} \rightarrow \mathbf{c}'$ in \mathbb{C} and a map $\gamma : x \rightarrow x'$ which covers the induced map $c_0 \rightarrow c'_0$.

One can check that the natural functor $\mathcal{X} \rightarrow \mathbb{C}$ is an opfibration, and it is easy to see that the assignment $(\mathbf{c}, \alpha : x \rightarrow y) \mapsto (\mathbf{c}, x)$ defines an equivalence over \mathbb{C} of opfibrations $\mathcal{X} \xrightarrow{\sim} h_{\mathbb{C}}^* \mathcal{E}$.

On the other hand, consider the assignment $(\mathbf{c}, \alpha : x \rightarrow y) \mapsto (\mathbf{c}, y)$. We claim that it defines a functor $\bar{T} : \mathcal{X} \rightarrow \mathbf{E}$ commuting with projections to \mathbb{C} . Let $(f, t) : (\mathbf{c}, \alpha : x \rightarrow y) \rightarrow (\mathbf{c}', \beta : x' \rightarrow y')$ be a map. In particular, we have the following diagram in \mathcal{E} :

$$\begin{array}{ccc} x & \xrightarrow{t} & x' \\ \alpha \downarrow & & \downarrow \beta \\ y & & y'. \end{array} \quad (3.3)$$

Suppose first that the map t is fiberwise. Then by opCartesian property there exists a map $t' : y \rightarrow y'$ rendering the diagram (3.3) commutative. Remembering the description of arrows in Definition 2.13, we define $\bar{T}(f, t) = (f, y \xrightarrow{t'} y' \xleftarrow{id} y')$; in other words, we view t' as a fiberwise map of \mathcal{E}^\top .

Next, if t is opCartesian, find an opCartesian map $k : y' \rightarrow z$ in \mathcal{E} covering $c'_m \rightarrow c_n$ (which is induced from $f : \mathbf{c} \rightarrow \mathbf{c}'$). The composition $k\beta t$ and α both project along $\mathcal{E} \rightarrow \mathbb{C}$ to the map $c_0 \rightarrow c_n = c_0 \rightarrow c'_0 \rightarrow c'_m \rightarrow c_n$, hence there is a (fiberwise) isomorphism $z \cong y$. This implies that the diagram (3.3) can be completed as

$$\begin{array}{ccc} x & \xrightarrow{t} & x' \\ \alpha \downarrow & & \downarrow \beta \\ y & \xleftarrow{t'} & y'. \end{array}$$

with all arrows opCartesian in \mathcal{E} . We put, again, $\bar{T}(f, t) = (f, y \xrightarrow{id} y \xleftarrow{t'} y')$, thus viewing t' as a Cartesian map of \mathcal{E}^\top . Any other case of (f, t) can be treated by reducing to these two cases.

Inverting the equivalence $\mathcal{X} \xrightarrow{\sim} h_{\mathcal{C}}^* \mathcal{E}$ and composing with \bar{T} , we obtain the desired functor $T : h_{\mathcal{C}}^* \mathcal{E} \rightarrow \mathbf{E}$, and one can verify its universal property. \square

Lemma 3.14. *Let $\mathcal{E} \rightarrow \mathcal{C}$ be a homotopical Δ -opfibration. Then $\mathbf{E} \rightarrow \mathcal{C}$ is a homotopical Δ -fibration.*

Proof. Clear. \square

3.2 Homotopical category of derived sections

Definition 3.15. Given an opfibration $\mathcal{E} \rightarrow \mathcal{C}$, its category of *presections* is the category

$$\mathbf{PSect}(\mathcal{C}, \mathcal{E}) := \mathbf{Sect}_{\mathcal{C}}(\mathcal{C}, \mathbf{E}).$$

Recall the functors $h_{\mathcal{C}}$ and T discussed before in Lemma 3.3 and Proposition 3.13.

Proposition 3.16. *The assignment $S \mapsto T \circ (h_{\mathcal{C}}^* S)$ defines a functor $i : \mathbf{Sect}(\mathcal{C}, \mathcal{E}) \rightarrow \mathbf{PSect}(\mathcal{C}, \mathcal{E})$. Its essential image consists of the presections sending the anchor maps $A_{\mathcal{C}}$ to Cartesian morphisms in \mathbf{E} .*

Proof. Note that for any anchor map $a : \mathbf{c}_{[n]} \rightarrow \mathbf{c}_{[k]}$ a map in $h_{\mathcal{C}}^* \mathcal{E}$ is opCartesian over a iff it is an isomorphism $x \xrightarrow{\sim} x$ in $\mathcal{E}(c_0)$. On one hand, the functor T sends such maps to Cartesian maps in \mathbf{E} ; on the other hand, the pullback section $h_{\mathcal{C}}^* S : \mathcal{C} \rightarrow h_{\mathcal{C}}^* \mathcal{E}$ sends anchor maps $A_{\mathcal{C}}$ precisely to identities in \mathcal{E} . Further details are then clear. \square

Remark 3.17. Consider an object $\mathbf{c}_{[n]} = c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} c_n$ of \mathcal{C} . Then $S \in \mathbf{Sect}(\mathcal{C}, \mathcal{E})$ is sent by the functor above to $i(S)$ such that $i(S)(\mathbf{c}_{[n]}) \cong (f_n \dots f_1)_! S(c_0)$ where $(f_n \dots f_1)_! : \mathcal{E}(c_0) \rightarrow \mathcal{E}(c_n) = \mathbf{E}(\mathbf{c}_{[n]})$ is a transition functor along the composition of f_i .

Assume now that $\mathcal{E} \rightarrow \mathcal{C}$ has a homotopical structure \mathcal{W} .

Definition 3.18. The *standard homotopical structure* on $\mathbf{PSect}(\mathcal{C}, \mathcal{E})$ is defined by the subcategory of those morphisms $A \rightarrow A'$ for which the map $A(\mathbf{c}_{[n]}) \rightarrow A'(\mathbf{c}_{[n]})$ is in \mathcal{W} for each $\mathbf{c}_{[n]} \in \mathcal{C}$.

We henceforth assume this homotopical structure whenever dealing with $\mathbf{PSect}(\mathcal{C}, \mathcal{E})$. We denote by $\mathbf{HoPSect}(\mathcal{C}, \mathcal{E})$ the corresponding localisation.

Definition 3.19. A presection $A : \mathcal{C} \rightarrow \mathbf{E}$ is a *derived section* iff A sends anchor maps to weakly Cartesian morphisms in \mathbf{E} .

We denote by $\mathbf{RSect}(\mathcal{C}, \mathcal{E})$ the full subcategory of $\mathbf{PSect}(\mathcal{C}, \mathcal{E})$ spanned by derived sections. We restrict the standard homotopical structure from $\mathbf{PSect}(\mathcal{C}, \mathcal{E})$ to $\mathbf{RSect}(\mathcal{C}, \mathcal{E})$ and denote by $\mathbf{HoRSect}(\mathcal{C}, \mathcal{E})$ the corresponding localisation.

4 The pushforward functor

Given a functor $F : \mathcal{D} \rightarrow \mathcal{C}$, there is an induced pullback morphism

$$\mathbb{F}^* : \text{PSect}(\mathcal{C}, \mathcal{E}) = \text{Sect}(\mathcal{C}, \mathbf{E}) \rightarrow \text{Sect}(\mathcal{D}, F^*\mathbf{E}) = \text{PSect}(\mathcal{D}, F^*\mathcal{E})$$

which restricts well to

$$\mathbb{F}^* : \mathbb{R}\text{Sect}(\mathcal{C}, \mathcal{E}) \rightarrow \mathbb{R}\text{Sect}(\mathcal{D}, F^*\mathcal{E})$$

and is moreover homotopical. It is natural to ask if such a functor may admit a homotopy adjoint.

In this section, we provide a partial answer to this question, by constructing

$$\mathbb{F}_! : \text{PSect}(\mathcal{D}, F^*\mathcal{E}) \rightarrow \text{PSect}(\mathcal{C}, \mathcal{E}),$$

a homotopical functor in the other direction, together with natural spans relating $\mathbb{F}_!\mathbb{F}^*$ and $\mathbb{F}^*\mathbb{F}_!$ with identity functors (Proposition 4.9). The functor $\mathbb{F}_!$ may be viewed as an almost left adjoint to the pullback functor \mathbb{F}^* . If $\mathbb{F}_!$ preserves derived sections, the spans mentioned above indeed give well-defined maps $\mathbb{F}_!\mathbb{F}^* \rightarrow id$ and $id \rightarrow \mathbb{F}^*\mathbb{F}_!$ on the level of homotopy categories of derived sections, satisfying a triangle identity. Even under a weaker assumption that the span relating $\mathbb{F}_!\mathbb{F}^*$ to the identity functor consists of weak equivalences, Corollary 4.10 implies that \mathbb{F}^* is full and faithful on homotopy level.

4.1 Main construction

Fix a functor $F : \mathcal{D} \rightarrow \mathcal{C}$ and a homotopical Δ -opfibration $\mathcal{E} \rightarrow \mathcal{C}$. Recall the diagram (3.1) for $G = id_{\mathcal{C}}$:

$$\begin{array}{ccc} & \mathbb{F} // \mathcal{C} & \\ \text{\scriptsize $pr_{\mathcal{D}}$} \swarrow & \downarrow \begin{smallmatrix} \Leftarrow \\ \Rightarrow \end{smallmatrix} & \searrow \text{\scriptsize $pr_{\mathcal{C}}$} \\ \mathcal{D} & \xrightarrow{\mathbb{F}} \mathcal{C} & \xleftarrow{id_{\mathcal{C}}} \mathcal{C}. \end{array} \quad (4.1)$$

The middle map is $pr_{\mathbb{F} // \mathcal{C}}$. This diagram gives us in particular the restriction morphism of Proposition 2.15

$$R_{\mathbb{F}} : (\mathbb{F}pr_{\mathcal{D}})^*\mathbf{E} \rightarrow pr_{\mathbb{F} // \mathcal{C}}^*\mathbf{E}. \quad (4.2)$$

This is a map of fibrations over $\mathbb{F} // \mathcal{C}$.

Next, we observe there are equivalences

$$\text{Sect}(\mathbb{F} // \mathcal{C}, pr_{\mathbb{F} // \mathcal{C}}^*\mathbf{E}) \xleftarrow{\sim} \text{Sect}(\mathbb{F} // \mathcal{C}, pr_{\mathcal{C}}^*\mathbf{E}) \xrightarrow{\sim} \text{Sect}(\mathcal{C}, \mathbf{E}^{\mathbb{F} // \mathcal{C}}) \quad (4.3)$$

where the right equivalence is provided by the first assessment of Lemma 2.18 (keep in mind that $pr_{\mathcal{C}}$ is a small opfibration). The left map comes from the equivalence

$$pr_{\mathcal{C}}^*\mathbf{E} \xrightarrow{\sim} pr_{\mathbb{F} // \mathcal{C}}^*\mathbf{E}$$

provided by Remark 3.12. We denote by

$$D_{\mathbb{F}} : \text{Sect}(\mathbb{F} // \mathcal{C}, pr_{\mathbb{F} // \mathcal{C}}^*\mathbf{E}) \xrightarrow{\sim} \text{Sect}(\mathcal{C}, \mathbf{E}^{\mathbb{F} // \mathcal{C}}) \quad (4.4)$$

the resulting equivalence constructed from (4.3).

There is a natural 'projection' functor Π over \mathbb{C} ,

$$\begin{array}{ccc} \mathbb{F} // \mathbb{C} & \xrightarrow{\Pi} & \Delta^{\text{op}} \times \mathbb{C} \\ & \searrow \text{pr}_{\mathbb{C}} & \swarrow \\ & \mathbb{C} & \end{array}$$

which acts as $(\mathbf{d}_{[n]}, \mathbf{c}_{[m]}, \alpha : F(d_n) \rightarrow c_0) \mapsto ([n], \mathbf{c}_{[m]})$. Exponentiating and taking sections, we obtain a functor $\Pi^* : \text{Sect}(\mathbb{C}, \mathbf{E}^{\Delta^{\text{op}}}) \rightarrow \text{Sect}(\mathbb{C}, \mathbf{E}^{\mathbb{F} // \mathbb{C}})$.

Proposition 4.1. *The functor*

$$\Pi^* : \text{Sect}(\mathbb{C}, \mathbf{E}^{\Delta^{\text{op}}}) \rightarrow \text{Sect}(\mathbb{C}, \mathbf{E}^{\mathbb{F} // \mathbb{C}})$$

admits a homotopical left adjoint

$$\Pi_! : \text{Sect}(\mathbb{C}, \mathbf{E}^{\mathbb{F} // \mathbb{C}}) \rightarrow \text{Sect}(\mathbb{C}, \mathbf{E}^{\Delta^{\text{op}}}) \quad (4.5)$$

Proof. See Proposition 2.19 for the construction of $\Pi_!$. To observe that it is homotopical, note that for each \mathbf{c} , the functor $\mathbb{F} // \mathbb{C}(\mathbf{c}) \rightarrow \Delta^{\text{op}}$ is a *discrete* opfibration, and the pushforward along it amounts to taking coproducts, which are homotopical. \square

Take a \mathcal{D} -presection $S : \mathbb{D} \rightarrow \mathbb{F}^* \mathbf{E}$. Then apply functors (4.2), (4.4) and (4.5) to obtain

$$B_{\bullet}(S) := \Pi_! D_{\mathbb{F}}(R_{\mathbb{F}} \circ \text{pr}_{\mathbb{D}}^* S) \in \text{Sect}(\mathbb{C}, \mathbf{E}^{\Delta}). \quad (4.6)$$

Lemma 3.14 implies that $\mathbf{E} \rightarrow \mathbb{C}$ is a homotopical Δ -fibration. Applying the realisation functor $|-|$ from Proposition 2.26, we get the following:

Definition 4.2. The *derived pushforward* of a presection $A : \mathbb{D} \rightarrow \mathbb{F}^* \mathbf{E}$ is defined as

$$\mathbb{F}_!(S) := |B_{\bullet}(S)| = |\Pi_! D_{\mathbb{F}}(R_{\mathbb{F}} \circ \text{pr}_{\mathbb{D}}^* S)|.$$

this defines a homotopical functor $\mathbb{F}_! : \text{PSect}(\mathcal{D}, \mathcal{E}) \rightarrow \text{PSect}(\mathbb{C}, \mathcal{E})$.

Remark 4.3. Over an object $\mathbf{c}_{[m]} = c_0 \xrightarrow{f_1} \dots \xrightarrow{f_m} c_m$, we have

$$B_n(S)(\mathbf{c}_{[m]}) = \coprod_{\mathbf{d}_{[n]}, \alpha : F(d_n) \rightarrow c_0} (f_m \dots f_1 \alpha)_! S(\mathbf{d}_{[n]})$$

where $(f_m \dots f_1 \alpha)_!$ is the transition functor $\mathcal{E}(F(d_n)) \rightarrow \mathcal{E}(c_m)$. This expression is very similar to the bar construction (cf. [5, 23]); the value $\mathbb{F}_! S(\mathbf{c}_{[m]})$ is just the realisation of this simplicial object $B_n(S)(\mathbf{c}_{[m]})$.

4.2 Unit and counit correspondences

Given a \mathcal{C} -presection $A : \mathbb{C} \rightarrow \mathbf{E}$, use $pr_{\mathbb{F}/\mathbb{C}}$ from the diagram (4.1) and functors (4.2), (4.4) and (4.5) to obtain

$$B_{\bullet}^{\mathbb{F}}(A) := \Pi_! D_{\mathbb{F}}(pr_{\mathbb{F}/\mathbb{C}}^* A) \in \text{Sect}(\mathbb{C}, \mathbf{E}^{\Delta}). \quad (4.7)$$

Denote by $A^{\mathbb{F}}$ the realisation of $B_{\bullet}^{\mathbb{F}}(A)$.

Remark 4.4. Again, one can see that explicitly

$$B_n^{\mathbb{F}}(A)(\mathbf{c}_{[m]}) = \coprod_{\mathbf{d}_{[n]}, \alpha: F(d_n) \rightarrow c_0} A(\mathbb{F}(\mathbf{d}_{[n]}) *^{\alpha} \mathbf{c}_{[m]}).$$

Lemma 4.5. *There is a natural (in A) correspondence in $\text{PSect}(\mathcal{C}, \mathcal{E})$*

$$\mathbb{F}_! \mathbb{F}^* A \leftarrow A^{\mathbb{F}} \rightarrow A$$

coming from the realisation of the correspondence of simplicial presections

$$B_{\bullet}(\mathbb{F}^* A) \leftarrow B_{\bullet}^{\mathbb{F}}(A) \rightarrow A$$

where the rightmost term is a constant simplicial object. When A is a derived section, the left morphisms in the correspondences above are weak equivalences.

Proof. First, the construction. Given a \mathcal{C} -presection $A : \mathbb{C} \rightarrow \mathbf{E}$, Proposition 2.15 and the left triangle of the diagram (4.1) gives us a map of $pr_{\mathbb{F}/\mathbb{C}}^* \mathbb{F}^* \mathbf{E}$ -sections over \mathbb{F}/\mathbb{C}

$$pr_{\mathbb{F}/\mathbb{C}}^* A \rightarrow R_{\mathbb{F}}(\mathbb{F} pr_{\mathbb{D}})^* A$$

which is an equivalence when A is a derived section. Indeed, over an object $(\mathbf{d}, \mathbf{c}, \alpha)$ of \mathbb{F}/\mathbb{C} the map looks like

$$A(\mathbb{F}(\mathbf{d}) *^{\alpha} \mathbf{c}) \rightarrow (f_n \dots f_1 \alpha)_! A(\mathbb{F}(\mathbf{d}))$$

with $\mathbf{c} = c_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} c_n$, and this map is an equivalence precisely because of the derived section condition for A . Applying the equivalence $D_{\mathbb{F}}$ of (4.4) and then $\Pi_!$ of (4.5), we get the map

$$B_{\bullet}^{\mathbb{F}}(A) \rightarrow B_{\bullet}(\mathbb{F}^* A)$$

between (4.6) and (4.7) which is again a weak equivalence when A is a derived section.

Proposition 2.15 and the right triangle of the diagram (4.1) give us a map

$$pr_{\mathbb{F}/\mathbb{C}}^* A \rightarrow pr_{\mathbb{C}}^* A$$

and we again apply $\Pi_! D_{\mathbb{F}}$. Observe that $\Pi_! D_{\mathbb{F}} pr_{\mathbb{C}}^* A$ is the following simplicial presection:

$$(\Pi_! D_{\mathbb{F}} pr_{\mathbb{C}}^* A)_n(\mathbf{c}) = \coprod_{\mathbf{d}_{[n]}, \alpha: F(d_n) \rightarrow c_0} A(\mathbf{c}) \cong N(F/c_0)(n) \otimes A(\mathbf{c}).$$

There is thus a natural map $\Pi_! D_{\mathbb{F}} pr_{\mathbb{C}}^* A \rightarrow A$ to the constant simplicial presection A .

The realisation of $\Pi_! D_{\mathbb{F}} pr_{\mathbb{C}}^* A$ is the presection given by the assignment $\mathbf{c} \mapsto N(F/c_0) \otimes A(\mathbf{c})$. On this level as well, we get the map

$$A^{\mathbb{F}} \rightarrow N(F/h_{\mathcal{C}}(-)) \otimes A \rightarrow A$$

which completes the construction. □

Lemma 4.6. *Let $A : \mathbb{C} \rightarrow \mathbf{E}$ be a derived section and $s : \mathbf{c}_{[n]} \rightarrow \mathbf{c}'_{[m]}$ be such a map in \mathbb{C} that its underlying map $s : [m] \rightarrow [n]$ in Δ is the surjective left inverse of the inclusion $i : [n] \rightarrow [m]$ of $[n]$ as last $n + 1$ objects of $[m]$. Then $A(s)$ is weakly Cartesian in \mathbf{E} .*

Proof. Clear. □

Lemma 4.7. *For $F = id_{\mathbb{C}}$ and a derived section $A : \mathbb{C} \rightarrow \mathbf{E}$ both morphisms in the span*

$$id_{\mathbb{C}!} id_{\mathbb{C}}^* A \leftarrow A^{id_{\mathbb{C}}} \rightarrow A$$

of Lemma 4.5 are weak equivalences.

Proof. Fix $\mathbf{c} \in \mathbb{C}$. In the case of the identity functor, we see that $B_{\bullet}^{id_{\mathbb{C}}} A(\mathbf{c})$ can be calculated as the realisation (cf. Definition 3.8) of the functor $X : \mathbb{C} // c_0 \rightarrow \mathbf{E}(\mathbf{c})$ defined by the assignment

$$X((\mathbf{c}'_{[k]}, \alpha : \mathbf{c}'_k \rightarrow c_0)) = A(\mathbf{c}'_{[k]} *^{\alpha} \mathbf{c}).$$

The category $\mathbb{C} // c_0$ is the simplicial replacement of the category \mathbb{C} / c_0 , and the latter has a terminal object. By Lemma 3.9, the natural map $A(c_0 *^{id_{c_0}} \mathbf{c}) = X(c_0) \rightarrow |\Pi_! X| = A^{id_{\mathbb{C}}}(\mathbf{c})$ is an equivalence.

There is also an equivalence $A(\mathbf{c}) \rightarrow A(c_0 *^{id_{c_0}} \mathbf{c})$ which comes from the degeneracy $\mathbf{c} \rightarrow c_0 *^{id_{c_0}} \mathbf{c}$ (cf. Lemma 4.6). One can then see that the composition

$$A(\mathbf{c}) \rightarrow A(c_0 *^{id_{c_0}} \mathbf{c}) \rightarrow A^{id_{\mathbb{C}}}(\mathbf{c}) \rightarrow A(\mathbf{c})$$

is the identity (it is such already on the level of corresponding simplicial objects; also note that the composition $\mathbf{c} \rightarrow c_0 *^{id_{c_0}} \mathbf{c} \rightarrow \mathbf{c}$ in \mathbb{C} is the identity $id_{\mathbb{C}}$). Thus the \mathbf{c} -th component of the map $A^{id_{\mathbb{C}}} \rightarrow A$ is an equivalence as a right inverse of an equivalence $A(\mathbf{c}) \rightarrow A(c_0 *^{id_{c_0}} \mathbf{c}) \rightarrow A^{id_{\mathbb{C}}}(\mathbf{c})$. □

Lemma 4.8. *For a functor $F : \mathbb{D} \rightarrow \mathbb{C}$ and a \mathbb{D} -presection A , there is a natural (in A) morphism*

$$id_{\mathbb{D}!} id_{\mathbb{D}}^* A \longrightarrow \mathbb{F}^* \mathbb{F}_! A.$$

Proof. By definition, $id_{\mathbb{D}!} id_{\mathbb{D}}^* A$ is the realisation of the simplicially valued presection X which at $d_0 \xrightarrow{g_1} \dots \xrightarrow{g_m} d_m$ takes the value

$$[n] \mapsto X_n = \coprod_{\substack{d'_0 \rightarrow \dots \rightarrow d'_n \\ \alpha : d'_n \rightarrow d_0}} (F(g_m \dots g_1 \alpha))_! A(\mathbf{d}'_{[n]}).$$

In the case when we calculate $\mathbb{F}^* \mathbb{F}_! A$ at $d_0 \xrightarrow{g_1} \dots \xrightarrow{g_m} d_m$, we have the following simplicial object Y :

$$[n] \mapsto Y_n = \coprod_{\substack{d'_0 \rightarrow \dots \rightarrow d'_n \\ \beta : F(d'_n) \rightarrow F(d_0)}} (F(g_m \dots g_1) \beta)_! A(\mathbf{d}'_{[n]}).$$

The assignment of $\alpha : d'_n \rightarrow d_0$ to $F\alpha : F(d'_n) \rightarrow F(d_0)$ induces the map of sets

$$\{d'_0 \rightarrow \dots \rightarrow d'_n, \alpha : d'_n \rightarrow d_0\} \rightarrow \{d'_0 \rightarrow \dots \rightarrow d'_n, \beta : F(d'_n) \rightarrow F(d_0)\} \quad (4.8)$$

and we obtain a map $X_n \rightarrow Y_n$ as X_n and Y_n are the coproducts indexed by the sets in (4.8). Varying $[n] \in \Delta$, we assemble a map $X \rightarrow Y$ of simplicial objects, which after realisations gives the map in question, $id_{\mathbb{D}!}id_{\mathbb{D}}^*A \rightarrow \mathbb{F}^*\mathbb{F}_!A$. \square

We finally prove the main proposition of this section:

Proposition 4.9. *Let $F : \mathcal{D} \rightarrow \mathcal{C}$, $A \in \mathbb{R}\text{Sect}(\mathcal{C}, \mathcal{E})$ and $R \in \mathbb{R}\text{Sect}(\mathcal{D}, F^*\mathcal{E})$.*

1. *There is a natural (in A) span of presections*

$$\mathbb{F}_!\mathbb{F}^*A \leftarrow A^{\mathbb{F}} \rightarrow A \quad (4.9)$$

which induces a natural transformation $\epsilon : \mathbb{F}_!\mathbb{F}^ \rightarrow id$ of functors $\text{Ho } \mathbb{R}\text{Sect}(\mathcal{C}, \mathcal{E}) \rightarrow \text{Ho } \mathbb{P}\text{Sect}(\mathcal{C}, \mathcal{E})$ (where id is the inclusion functor).*

2. *There is a natural (in R) sequence of morphisms*

$$R \leftarrow R^{id_{\mathbb{D}}} \rightarrow id_{\mathbb{D}!}id_{\mathbb{D}}^*R \rightarrow \mathbb{F}^*\mathbb{F}_!R \quad (4.10)$$

which induces a natural transformation $\eta : id \rightarrow \mathbb{F}^\mathbb{F}_!$ of functors $\text{Ho } \mathbb{R}\text{Sect}(\mathcal{D}, F^*\mathcal{E}) \rightarrow \text{Ho } \mathbb{P}\text{Sect}(\mathcal{D}, F^*\mathcal{E})$.*

3. (Triangle identity) *For each $A \in \text{Ho } \mathbb{R}\text{Sect}(\mathcal{C}, \mathcal{E})$, the composition in $\text{Ho } \mathbb{P}\text{Sect}(\mathcal{D}, F^*\mathcal{E})$*

$$\mathbb{F}^*A \xrightarrow{\eta_{\mathbb{F}^*A}} \mathbb{F}^*\mathbb{F}_!\mathbb{F}^*A \xrightarrow{\mathbb{F}^*\epsilon_A} \mathbb{F}^*A \quad (4.11)$$

is the identity.

Proof. We proved the first two claims in the preceding lemmas. Only the triangle identity remains. Using the correspondences obtained before, we write a string of morphisms

$$\mathbb{F}^*A \xleftarrow{\sim} (\mathbb{F}^*A)^{id_{\mathbb{D}}} \xrightarrow{\sim} id_{\mathbb{D}!}id_{\mathbb{D}}^*\mathbb{F}^*A \rightarrow \mathbb{F}^*\mathbb{F}_!\mathbb{F}^*A \xleftarrow{\sim} \mathbb{F}^*(A^{\mathbb{F}}) \rightarrow \mathbb{F}^*A$$

with all the weak equivalences drawn as $\xrightarrow{\sim}$ or $\xleftarrow{\sim}$. We can redraw this sequence, obtaining the (potentially non-commutative) diagram

$$\begin{array}{ccc} & (\mathbb{F}^*A)^{id_{\mathbb{D}}} & \xrightarrow{\sim} id_{\mathbb{D}!}id_{\mathbb{D}}^*\mathbb{F}^*A \\ & \searrow \sim & \downarrow \\ \mathbb{F}^*A & & \mathbb{F}^*\mathbb{F}_!\mathbb{F}^*A \\ & \nearrow \sim & \uparrow \sim \\ & \mathbb{F}^*(A^{\mathbb{F}}) & \xrightarrow{\sim} \mathbb{F}^*\mathbb{F}_!\mathbb{F}^*A \end{array}$$

The third claim is then equivalent to the commutativity of this diagram. We proceed as follows: writing down in components the simplicial object used to obtain $(\mathbb{F}^* A)^{id_{\mathbb{D}}}$, we see

$$(\mathbb{F}^* A)^{id_{\mathbb{D}}} \longleftrightarrow B_n^{id_{\mathbb{D}}}(\mathbb{F}^* A)(\mathbf{d}_{[m]}) = \coprod_{\mathbf{d}'_{[n]}, \alpha: d'_n \rightarrow d_0} A(\mathbb{F}(\mathbf{d}'_{[n]} *^\alpha \mathbf{d}_{[m]})).$$

In the same way,

$$\mathbb{F}^*(A^{\mathbb{F}}) \longleftrightarrow (\mathbb{F}^* B_n^{\mathbb{F}}(A))(\mathbf{d}_{[m]}) = \coprod_{\mathbf{d}'_{[n]}, \beta: F(d'_n) \rightarrow F(d_0)} A(\mathbb{F}(\mathbf{d}'_{[n]}) *^\beta \mathbb{F}(\mathbf{d}_{[m]})).$$

Assigning $\alpha \mapsto F(\alpha)$, we see that there is a natural in A map $(\mathbb{F}^* A)^{id_{\mathbb{D}}} \rightarrow \mathbb{F}^*(A^{\mathbb{F}})$. Moreover, a comparison with the construction of Lemma 4.8 reveals that in the resulting diagram

$$\begin{array}{ccc} (\mathbb{F}^* A)^{id_{\mathbb{D}}} & \xrightarrow{\sim} & id_{\mathbb{D}!} id_{\mathbb{D}}^* \mathbb{F}^* A \\ \swarrow \sim & \downarrow & \downarrow \\ \mathbb{F}^* A & & \mathbb{F}^* \mathbb{F}_! \mathbb{F}^* A \\ & \downarrow & \\ & \mathbb{F}^*(A^{\mathbb{F}}) & \xrightarrow{\sim} \mathbb{F}^* \mathbb{F}_! \mathbb{F}^* A \end{array}$$

both the left-hand triangle and the right-hand square commute. \square

Corollary 4.10. *Assume that for a functor $F : \mathcal{D} \rightarrow \mathcal{C}$, both maps in the span (4.9) are weak equivalences. Then $\mathbb{F}^* : \text{Ho } \mathbb{R}\text{Sect}(\mathcal{C}, \mathcal{E}) \rightarrow \text{Ho } \mathbb{R}\text{Sect}(\mathcal{D}, F^* \mathcal{E})$ is full and faithful.*

Proof. This result can be proven as a particular case of the following categorical result:

Let $f : \mathcal{M} \rightleftarrows \mathcal{N} : u$ be two functors, $i : \mathcal{N}_0 \subset \mathcal{N}$ and $j : \mathcal{M}_0 \subset \mathcal{M}$ are full subcategories such that ui is contained in \mathcal{M}_0 . In other words, there is a functor $u_0 : \mathcal{N}_0 \rightarrow \mathcal{M}_0$ with $ui = ju_0$. Suppose furthermore that there are natural transformations $\epsilon : fui \xrightarrow{\sim} i$ and $\eta : j \rightarrow u f j$ defined over \mathcal{N}_0 and \mathcal{M}_0 respectively such that the triangle identity is satisfied: $ui = ju_0 \rightarrow u f ju_0 = u f ui \rightarrow ui$ is the identity. Then u_0 , or equivalently ui , is full and faithful.

In turn, the categorical result is proven as follows. The functoriality of u_0 supplies us with maps $u(x, y) : \mathcal{N}_0(x, y) \rightarrow \mathcal{M}_0(u_0 x, u_0 y) = \mathcal{M}(uix, uiy)$. Given a map $\alpha : ux \rightarrow uy$, we define $v(x, y)\alpha$ to be the map fitting in the commutative square

$$\begin{array}{ccc} fui x & \xrightarrow{f\alpha} & fui y \\ \epsilon_x \downarrow \sim & & \sim \downarrow \epsilon_y \\ ix & \xrightarrow{v(x, y)\alpha} & iy \end{array}$$

(here we use that i is a full and faithful inclusion). This defines the map $v(x, y) : \mathcal{M}(uix, uiy) \rightarrow \mathcal{N}_0(x, y)$ which is inverse to $u(x, y)$. \square

Note in particular that in the situation like above, for $A \in \mathbb{R}\text{Sect}(\mathcal{C}, \mathcal{E})$, $\mathbb{F}_! \mathbb{F}^* A$ is again a derived section.

5 Case of a resolution

In this section, we study the functors of \mathbb{F}^* and $\mathbb{F}_!$ for a particular class of functors F , which we call resolutions:

Definition 5.1. A functor $F : \mathcal{D} \rightarrow \mathcal{C}$ is a *resolution* if it is an opfibration and each fiber $\mathcal{D}(c)$ is contractible (that is, its nerve $N\mathcal{D}(c)$ is contractible).

Remark 5.2. Resolutions $F : \mathcal{D} \rightarrow \mathcal{C}$ should properly be called partial resolutions since we will not make a claim about the smoothness of any category of derived sections defined for \mathcal{D} , in any possible sense (for what it may mean in the setting of enhanced triangulated or dg-categories, the reader is invited to consult [26]).

Denote by $\mathbb{D}(c)$ the simplicial replacement of $\mathcal{D}(c)$.

Definition 5.3. Let $F : \mathcal{D} \rightarrow \mathcal{C}$ be a resolution. A presection $A : \mathbb{D} \rightarrow \mathbb{F}^*\mathbf{E}$ is *locally constant* if for any fibre $\mathcal{D}(c)$ over $c \in \mathcal{C}$, the composite functor

$$\mathbb{D}(c) \rightarrow \mathbb{D} \xrightarrow{A} \mathbf{E}(c) = \mathcal{E}(c)$$

sends all morphisms of the domain to weak equivalences. A derived presection is locally constant if it is locally constant as a presection.

We denote by $\mathbf{PSect}(\mathcal{D}, \mathcal{E})_{lc}$ and $\mathbf{RSect}(\mathcal{D}, \mathcal{E})_{lc}$ the corresponding full homotopical subcategories of locally constant (pre)sections. It is clear that any (pre)section of the form \mathbb{F}^*A is locally constant.

When F is a resolution, we can prove two general results concerning \mathbb{F}^* . Here is the first result:

Theorem 5.4. *Let $\mathcal{E} \rightarrow \mathcal{C}$ be a homotopical Δ -opfibration and $F : \mathcal{D} \rightarrow \mathcal{C}$ be a resolution (Definition 5.1). Then after passing to localisations, the pullback functor $\mathbb{F}^* : \mathbf{Ho} \mathbf{RSect}(\mathcal{C}, \mathcal{E}) \rightarrow \mathbf{Ho} \mathbf{RSect}(\mathcal{D}, \mathcal{E})$ is full and faithful.*

The proof of this theorem relies on the machinery of pushforwards considered in the previous chapter leading to Corollary 4.10 and the additional manipulations are similar in the spirit to the proof of Cofinality Theorem in [5]. Indeed, we shall prove that $\mathbb{F}_!S(\mathbf{c}_{[n]})$ which is calculated as the realisation of

$$[n] \mapsto B_n(S)(\mathbf{c}_{[n]}) = \coprod_{\mathbf{d}_{[n]} \subset \mathbb{D}, \alpha: F(d_n) \rightarrow c_0} (f_m \dots f_1 \alpha)_! S(\mathbf{d}_{[n]})$$

can be also calculated, up to a coherent zigzag of equivalences, as the realisation of

$$[n] \mapsto \coprod_{\mathbf{d}_{[n]} \subset \mathbb{D}(c_0)} (f_m \dots f_1)_! S(\mathbf{d}_{[n]}).$$

When $S = \mathbb{F}^*A$ for a derived section $A : \mathcal{C} \rightarrow \mathbf{E}$, the second realisation is seen to be equivalent to $A(\mathbf{c}_{[n]})$. The difficulty of the proof comes mostly from the complexity of objects involved, and the necessity to make sure that the aforementioned zigzag of equivalences is (equivalent, for a fixed \mathbf{c} , to) the counit correspondence (4.9) of Proposition 4.9.

We can also characterise the homotopical essential image of \mathbb{F}^* . Unfortunately, we only know how to do it for F -special (cf. Definition 5.15) homotopical Δ -opfibrations:

Theorem 5.5. *Let $F : \mathcal{D} \rightarrow \mathcal{C}$ be a resolution and $\mathcal{E} \rightarrow \mathcal{C}$ be a F -special homotopical Δ -opfibration. Then the functor $\mathbb{F}^* : \mathrm{Ho} \mathbb{R}\mathrm{Sect}(\mathcal{C}, \mathcal{E}) \rightarrow \mathrm{Ho} \mathbb{R}\mathrm{Sect}(\mathcal{D}, \mathcal{E})_{lc}$ is an equivalence.*

However, while Definition 5.15 is fairly technical, the condition of speciality is satisfied when, for example, each fiber of the fibration $\mathbf{E} \rightarrow \mathcal{C}$ is a model category, and taking a realisation of any simplicial object $X : \Delta^{\mathrm{op}} \rightarrow \mathcal{E}(c)$ amounts to calculating its homotopy colimit (see [6] for the discussion of locally constant functors in this setting). This includes examples like \mathbf{DVect}_k or any other opfibrations which describe families of model categories with a reasonable notion of geometric realisation leading up to a Δ -structure.

5.1 Fullness and faithfulness

The main result of this section which we use to prove Theorem 5.4 is the following proposition:

Proposition 5.6. *Let $F : \mathcal{D} \rightarrow \mathcal{C}$ be a resolution. Then for any homotopical Δ -opfibration $\mathcal{E} \rightarrow \mathcal{C}$, the counit transformation*

$$\epsilon : \mathbb{F}_! \mathbb{F}^* A \rightarrow id_{\mathrm{Ho} \mathbb{P}\mathrm{Sect}(\mathcal{C}, \mathcal{E})} A$$

is an isomorphism in $\mathrm{Ho} \mathbb{P}\mathrm{Sect}(\mathcal{C}, \mathcal{E})$ for any derived section A .

The proof will be carried out in several steps. Note that for an opfibration $F : \mathcal{D} \rightarrow \mathcal{C}$ and an object $c \in \mathcal{C}$, we can take two categories F/c , the comma category of F and c (viewed as a functor $[0] \rightarrow \mathcal{C}$), and $\mathcal{D}(c)$, the fiber of F at c . There is a functor which sends $d \in \mathcal{D}(c)$ to $(d, id_c : F(d) \xrightarrow{\cong} c) \in F/c$ and it has a left adjoint given by choosing, for each object $(d, f : F(d) \rightarrow c) \in F/c$, an opCartesian morphism $d \rightarrow f_! d$ covering f . A similar pattern occurs a few times in this section, and this motivates us to introduce the following technical notion:

Definition 5.7. For a category \mathcal{D} , a functor $F : \mathcal{D} \rightarrow \mathcal{C}$ and an object $c \in \mathcal{C}$, a (F, c) -transition structure consists of

1. two categories I, J and functors $I : I \rightarrow \mathcal{D}$, $J : J \rightarrow \mathcal{D}$,
2. a functor $R : J \rightarrow I$ in \mathbf{Cat}/\mathcal{D} .

These data are subject to the following conditions:

- R admits a left adjoint L in \mathbf{Cat} ,
- J maps J to the fiber $\mathcal{D}(c)$, so that FJ factors through c .

In the notation of this definition, we sometimes write (I, J, R) to denote a given (F, c) -transition structure.

Example 5.8. The transition structures of importance for us are the following:

1. For an opfibration $F : \mathcal{D} \rightarrow \mathcal{C}$ and an object $c \in \mathcal{C}$, there is a (F, c) -transition structure given by $I = F/c$ and $J = \mathcal{D}(c)$ outlined just before Definition 5.7.

2. If $F : \mathcal{D} \rightarrow \mathcal{C}$ is an opfibration and $d \in \mathcal{D}$, one can have the following $(F, F(d))$ -transition structure: $I = \mathcal{D}/d$ and $J = \mathcal{D}(F(d))/d$. The right adjoint R is the evident inclusion; the left adjoint L is given by factoring any morphism $d' \rightarrow d$ as 'opCartesian followed by fiberwise' pair of morphisms.
3. Any (F, c) structure (I, \mathcal{J}, R) induces a $(F \circ I, c)$ structure (id_I, R, R) with the same right adjoint R . Thus the first example gives us a (F_c, c) -structure where $F_c : \mathcal{D}/c \rightarrow \mathcal{C}$ is the functor $(d, f : Fd \rightarrow c) \mapsto Fd$. For this structure, $I = \mathcal{D}/c$, $J = \mathcal{D}(c)$ and R acts in the same way as before.

Remark 5.9. Consider the unit map $\eta(i) : i \rightarrow RLi$ for any $i \in I$. Apply $F \circ I$ to this map and obtain $\bar{\eta}(i) : FI(i) \rightarrow c$. For any opfibration $\mathcal{E} \rightarrow \mathcal{C}$ we then have a well-defined 'restriction' functor

$$R_c : I^* \mathbb{F}^* \mathbf{E} \rightarrow \mathbf{E}(c) = \mathcal{E}(c)$$

where we denote by the same letter I the induced functor $\mathbb{I} \rightarrow \mathbb{D}$. Concretely, this functor sends $(\mathbf{i}_{[n]}, x \in \mathcal{E}(FI(i_n)))$ to $\bar{\eta}(i)_! x$, using a (chosen) opCartesian lift $x \rightarrow \bar{\eta}(i)_! x$ in \mathcal{E} covering $\bar{\eta}(i) : FI(i_n) \rightarrow c$.

Construction 5.10. Assume given $c_0 \rightarrow \dots \rightarrow c_n = \mathbf{c} \in \mathcal{C}$. Denote by $\mathbf{c}_!$ the natural transition functor

$$\mathbf{c}_! : \mathbf{E}(c_0) \cong \mathcal{E}(c_0) \rightarrow \mathcal{E}(c_n) \cong \mathbf{E}(c_n).$$

Consider also the simplicial comma object (Definition 3.7) $\mathbb{I} // \mathbb{R}$, where $\mathbb{R} : \mathbb{J} \rightarrow \mathbb{I}$ is the simplicial replacement of R . Using the diagram (3.1) and postcomposing with functors to \mathbb{D} we obtain a new diagram

$$\begin{array}{ccc} & \mathbb{I} // \mathbb{R} & \\ pr_{\mathbb{I}} \swarrow & \Downarrow & \searrow pr_{\mathbb{J}} \\ \mathbb{I} & \xrightarrow{I} \mathbb{D} & \xleftarrow{J} \mathbb{J} \end{array} \quad (5.1)$$

and we henceforth denote the middle map again by $pr_{\mathbb{I} // \mathbb{R}}$.

For $B \in \text{PSect}(\mathcal{D}, \mathcal{E})$ and a given (F, c_0) -structure, we get the following diagram

$$\begin{array}{ccc} & \mathbb{I} // \mathbb{R} & \\ pr_{\mathbb{I}} \swarrow & \Downarrow & \searrow pr_{\mathbb{J}} \\ \mathbb{I} & \xrightarrow{\mathbf{c}_! R_{c_0} I^* B} \mathbf{E}(\mathbf{c}) & \xleftarrow{\mathbf{c}_! J^* B} \mathbb{J} \end{array}$$

with the middle map $\mathbf{c}_! pr_{\mathbb{I} // \mathbb{R}}^* B$. Thus we have the span

$$pr_{\mathbb{I}}^* \mathbf{c}_! R_{c_0} I^* B \longleftarrow \mathbf{c}_! pr_{\mathbb{I} // \mathbb{R}}^* B \longrightarrow pr_{\mathbb{J}}^* \mathbf{c}_! J^* B.$$

Pushing this forward to the span given by projections,

$$\Delta^{\text{op}} \xleftarrow{\pi_1} \Delta^{\text{op}} \times \Delta^{\text{op}} \xrightarrow{\pi_2} \Delta^{\text{op}},$$

we obtain a span of bisimplicial objects in $\mathbf{E}(c)$:

$$\pi_1^* \Pi(\mathbf{c}_! R_{c_0} I^* B) \longleftarrow \Pi \mathbf{c}_! pr_{\mathbb{I}/\mathbb{R}}^* B \longrightarrow \pi_2^* \Pi(\mathbf{c}_! \mathcal{J}^* B). \quad (5.2)$$

Here Π is the pushforward functor, simplicial (along $\mathbb{I} \rightarrow \Delta^{\text{op}}$ and same for \mathbb{J}) or bisimplicial (along $\mathbb{I}/\mathbb{R} \rightarrow \Delta^{\text{op}} \times \Delta^{\text{op}}$). We implicitly used the Beck-Chevalley morphisms, such as $\Pi pr_{\mathbb{I}}^* \rightarrow \pi_1^* \Pi$, for pullbacks and pushforwards; they arise from commutative squares like

$$\begin{array}{ccc} \mathbb{I}/\mathbb{R} & \xrightarrow{pr_{\mathbb{I}}} & \mathbb{I} \\ \downarrow & & \downarrow \\ \Delta^{\text{op}} \times \Delta^{\text{op}} & \xrightarrow{\pi_1} & \Delta^{\text{op}} \end{array}$$

by taking associated pullback functors on functor categories and then replacing some of them by left adjoints.

Remark 5.11. Let us write the terms of the span (5.2) explicitly. For $\mathbf{c} = c_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} c_n$, we find that

$$\Pi(\mathbf{c}_! R_{c_0} I^* B)_m = \coprod_{\mathbf{i}_{[m]}} (f_n \dots f_1 \bar{\eta}(i_m))_! B(I \mathbf{i}_{[m]}),$$

where $\bar{\eta}(i_m)$ is induced from the unit of $L \dashv R$ (Remark 5.9). Next,

$$\Pi(\mathbf{c}_! \mathcal{J}^* B)_l = \coprod_{\mathbf{j}_{[l]}} (f_n \dots f_1)_! B(\mathcal{J} \mathbf{j}_{[l]}),$$

and, finally,

$$(\Pi \mathbf{c}_! pr_{\mathbb{I}/\mathbb{R}}^* B)_{ml} = \coprod_{\mathbf{i}_{[m]}, \mathbf{j}_{[l]}, \alpha: i_m \rightarrow R j_0} (f_n \dots f_1)_! B(I \mathbf{i}_{[m]}) *^{I\alpha} \mathcal{J}(\mathbf{j}_{[l]}).$$

Proposition 5.12. For $c_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} c_n = \mathbf{c} \in \mathbb{C}$, a $(F : \mathcal{D} \rightarrow \mathcal{C}, c_0)$ -transition structure (I, \mathcal{J}, R) , and any $B \in \mathbb{R}\text{Sect}(\mathcal{D}, F^* \mathcal{E})$ there is a natural (in B) span of weak equivalences in $\mathbf{E}(\mathbf{c})$

$$|\Pi(\mathbf{c}_! R_{c_0} I^* B)| \longleftarrow |\Pi \mathbf{c}_! pr_{\mathbb{I}/\mathbb{R}}^* B| \longrightarrow |\Pi(\mathbf{c}_! \mathcal{J}^* B)| \quad (5.3)$$

which comes from a natural (in B) span (5.2) of bisimplicial objects in $\mathbf{E}(\mathbf{c})$.

Proof. We need to prove that after realisations, both arrows become equivalences. Consider the bisimplicial object

$$\Pi \pi_1^*(\mathbf{c}_! R_{c_0} I^* B)_{ml} = \coprod_{\mathbf{i}_{[m]}, \mathbf{j}_{[l]}, \alpha: i_m \rightarrow R j_0} (f_n \dots f_1 \bar{\eta}(i_m))_! B(I \mathbf{i}_{[m]})$$

Our left hand side map in (5.2) passes through this object, as it is equal to the composition

$$\Pi \mathbf{c}_! pr_{\mathbb{I}/\mathbb{R}}^* B \rightarrow \Pi \pi_1^*(\mathbf{c}_! R_{c_0} I^* B) \rightarrow \pi_1^* \Pi(\mathbf{c}_! R_{c_0} I^* B). \quad (5.4)$$

Writing down the simplicial objects explicitly, we see that the first map in (5.4) arises from the action of B on anchor maps and is a termwise weak equivalence of bisimplicial objects

because B is a derived section. Realising the second map $\Pi\pi_1^*(\mathbf{c}!R_{c_0}I^*B) \rightarrow \pi_1^*\Pi(\mathbf{c}!R_{c_0}I^*B)$ in (5.4) along the second simplicial argument, we obtain a map in $\Delta^{\text{op}}\mathbf{E}(\mathbf{c})$, whose m -th component is

$$|\Pi\pi_1^*(\mathbf{c}!R_{c_0}I^*B)|_m \cong N(i_m \backslash \mathbf{R}) \otimes \Pi(\mathbf{c}!R_{c_0}I^*B)_m \rightarrow \Pi(\mathbf{c}!R_{c_0}I^*B)_m. \quad (5.5)$$

Observe that because of the adjunction $L \dashv R$, the category $i_m \backslash \mathbf{R} = (\mathbf{R}^{\text{op}}/i_m)^{\text{op}}$ has an initial object (the unit at i_m) and is thus contractible, so the map (5.5) and thus (5.4) and the left-hand side map in (5.2) are all weak equivalences.

We now have to prove that the right-hand side map $\Pi\mathbf{c}!pr_{\mathbb{I}/\mathbb{R}}^*B \rightarrow \pi_2^*\Pi(\mathbf{c}!\mathcal{I}^*B)$ in (5.2) becomes an equivalence after realisations. For each fixed $\mathbf{j}_{[l]}$, we have a map of simplicial objects, written in components as

$$\coprod_{\mathbf{i}_{[m]}, \alpha: i_m \rightarrow Rj_0} (f_n \dots f_1)_! B(I(\mathbf{i}_{[m]}) *^{I\alpha} \mathcal{I}(\mathbf{j}_{[l]})) \rightarrow (f_n \dots f_1)_! B(\mathcal{I}(\mathbf{j}_{[l]})); \quad (5.6)$$

because L/j_0 has a terminal object, Lemma 3.9 and Lemma 4.6 imply that the map (5.6) is a weak equivalence after being realised. We conclude that the morphism

$$|\Pi\mathbf{c}!pr_{\mathbb{I}/\mathbb{R}}^*B| \rightarrow |\pi_2^*\Pi(\mathbf{c}!\mathcal{I}^*B)| \cong |\Pi(\mathbf{c}!\mathcal{I}^*B)|$$

is an equivalence of simplicial objects in $\mathbf{E}(\mathbf{c})$, where we took the realisation of bisimplicial objects along the first argument. Proposition 1.26 then implies that the double realisation

$$||\Pi\mathbf{c}!pr_{\mathbb{I}/\mathbb{R}}^*B|| \rightarrow ||\pi_2^*\Pi(\mathbf{c}!\mathcal{I}^*B)|| \cong ||\Pi(\mathbf{c}!\mathcal{I}^*B)||,$$

taken in any order, is a weak equivalence. \square

We are now ready to prove Proposition 5.6. Fix $\mathbf{c}_{[n]} \in \mathbb{C}$. For $A \in \mathbb{R}\text{Sect}(\mathbb{C}, \mathcal{E})$ there are functors $A_{\mathbf{c}}^{\text{aug}}$ and $A_{\mathbf{c}}$ (cf. the proof of Lemma 4.5):

$$\begin{aligned} A_{\mathbf{c}}^{\text{aug}} : \mathbb{F}/c_0 &\rightarrow \mathbf{E}(\mathbf{c}_{[n]}), \quad (\mathbf{d}_{[m]}, \alpha : Fd_m \rightarrow c_0) \mapsto A(\mathbb{F}(\mathbf{d}_{[m]}) *^{\alpha} \mathbf{c}_{[n]}), \\ A_{\mathbf{c}} : \mathbb{F}/c_0 &\rightarrow \mathbf{E}(\mathbf{c}_{[n]}), \quad (\mathbf{d}_{[m]}, \alpha : Fd_m \rightarrow c_0) \mapsto A(\mathbf{c}_{[n]}). \end{aligned}$$

There is an obvious natural transformation $A_{\mathbf{c}}^{\text{aug}} \rightarrow A_{\mathbf{c}}$. Pushing it forward to Δ^{op} and realising gives us a map $A^{\mathbb{F}}(\mathbf{c}) \rightarrow N(F/c_0) \otimes A(\mathbf{c})$ so that the obvious composition

$$A^{\mathbb{F}}(\mathbf{c}) \rightarrow N(F/c_0) \otimes A(\mathbf{c}) \rightarrow A(\mathbf{c})$$

is the \mathbf{c} -th component of the right-hand map of the counit correspondence (4.9).

Lemma 5.13. *The morphism $N(F/c_0) \otimes A(\mathbf{c}) \rightarrow A(\mathbf{c})$ is a weak equivalence.*

Proof. There is an adjunction $F/c_0 \rightleftharpoons \mathcal{D}(c_0)$ and $\mathcal{D}(c_0)$ is contractible, thus F/c_0 is contractible as well because adjunctions of categories are known to induce homotopy equivalences between the associated nerves [22]. \square

Now recall Example 5.8(3) where we work over F/c_0 , with $I = F/c_0$, $J = \mathcal{D}(c_0)$ and $R : \mathcal{D}(c_0) \rightarrow F/c_0$ being the evident functor. Also take the trivial opfibration $\mathcal{E}(c_n) \times \mathbb{C} \rightarrow \mathbb{C}$. Both $A_{\mathbf{c}}$ and $A_{\mathbf{c}}^{\text{aug}}$ are then sections over \mathbb{F}/c_0 of the trivial fibration $\mathbf{E}(\mathbf{c}_{[n]}) \times \mathbb{F}/c_0 \rightarrow \mathbb{F}/c_0$.

Lemma 5.14. *The map $A^{\mathbb{F}}(\mathbf{c}) \rightarrow N(F/c_0) \otimes A(\mathbf{c})$ is a weak equivalence.*

Proof. The obvious natural transformation $A_{\mathbf{c}}^{aug} \rightarrow A_{\mathbf{c}}$, when plugged in the left hand side of the span (5.3) for the transition structure of the Example 5.8(3), gives us the map in question, $A^{\mathbb{F}}(\mathbf{c}) \rightarrow N(F/c_0) \otimes A(\mathbf{c})$. The right-hand side of span (5.3) gives the map

$$|\Pi(R^*A_{\mathbf{c}}^{aug})| \rightarrow N(\mathcal{D}(c_0)) \otimes A(\mathbf{c}) \quad (5.7)$$

so by Proposition 5.12 we are done if the map (5.7) is a weak equivalence. Observe however that

$$\Pi(R^*A_{\mathbf{c}}^{aug})_m = \coprod_{\mathbf{d}_{[m]} \in \mathbb{D}(c_0)} A(\mathbb{F}(\mathbf{d}_{[m]}) * \mathbf{c}) = \coprod_{\mathbf{d} \in \mathbb{D}(c_0)} A(id_{c_0}^m * \mathbf{c})$$

with $id_{c_0}^m$ being the degenerate m -simplex $c_0 \xrightarrow{id_{c_0}} \dots \xrightarrow{id_{c_0}} c_0$. Because A is a derived section, Lemma 4.6 implies that the obvious map $A(id_{c_0}^m * \mathbf{c}) \rightarrow A(\mathbf{c})$ is a weak equivalence, so that

$$\Pi(R^*A_{\mathbf{c}}^{aug})_m \rightarrow N(\mathcal{D}(c_0))_m \otimes A(\mathbf{c}) = \Pi(R^*A_{\mathbf{c}})_m$$

is a weak equivalence as well. \square

Varying \mathbf{c} , we obtain the proof of Proposition 5.6. With Corollary 4.10, we get that \mathbb{F}^* is fully faithful on homotopy level, which is exactly the contents of Theorem 5.4.

5.2 Essential surjectivity

Our second main result, Theorem 5.5, needs a technical condition of speciality. To state it, we need to define a few auxiliary things. First, take any opfibration $F : \mathcal{D} \rightarrow \mathcal{C}$. When F is viewed as a functor $\mathcal{C} \rightarrow \mathbf{Cat}$, we can compose it with the endofunctor $\mathbf{Cat} \rightarrow \mathbf{Cat}$ which is the simplicial replacement functor. On the level of opfibrations, define the category⁶ $\mathbb{O}_{\mathcal{C}}(\mathcal{D})$ as follows. An object of $\mathbb{O}_{\mathcal{C}}(\mathcal{D})$ is an object $c \in \mathcal{C}$ and $\mathbf{d} \in \mathbb{D}(c)$. A morphism $(c, \mathbf{d}_{[n]}) \rightarrow (c', \mathbf{d}'_{[m]})$ consists of a map $f : c \rightarrow c'$ and an equivalence class of pairs (β, γ) where

- $\beta : \mathbf{d}_{[n]} \Rightarrow \mathbf{d}_{[n]}^0$ is some natural transformation in $Fun([n], \mathcal{D})$ with domain $\mathbf{d}_{[n]}$ and so that each $\beta_i : d_i \rightarrow d_i^0$ is an opCartesian morphism in \mathcal{D} lying over $f : c \rightarrow c'$,
- $\gamma : \mathbf{d}_{[n]}^0 \rightarrow \mathbf{d}'_{[m]}$ is a morphism in $\mathbb{D}(c')$,
- and the equivalence relation is as follows. Two pairs $(\beta^0 : \mathbf{d}_{[n]} \Rightarrow \mathbf{d}_{[n]}^0, \gamma^0 : \mathbf{d}_{[n]}^0 \rightarrow \mathbf{d}'_{[m]})$ and $(\beta^1 : \mathbf{d}_{[n]} \Rightarrow \mathbf{d}_{[n]}^1, \gamma^1 : \mathbf{d}_{[n]}^1 \rightarrow \mathbf{d}'_{[m]})$ are equivalent if, after applying the functor $\pi : \mathbb{D}(c') \rightarrow \Delta^{\text{op}}$, we have that $\pi\gamma^0 = \pi\gamma^1$.

In all, we obtain an opfibration $\mathbb{O}_{\mathcal{C}}(\mathcal{D}) \rightarrow \mathcal{C}$ whose fibers are $\mathbb{D}(c)$ and whose transition functors are given by the simplicial replacements of f , the transition functors of $F : \mathcal{D} \rightarrow \mathcal{C}$ associated to $f : c \rightarrow c'$.

For any opfibration $F : \mathcal{D} \rightarrow \mathcal{C}$, denote by $F^*F : F^*\mathcal{D} \rightarrow \mathcal{D}$ the pullback opfibration of F along F . Then from F^*F we obtain the opfibration $\mathbb{O}_{\mathcal{D}}(F^*\mathcal{D}) \rightarrow \mathcal{D}$ constructed as

⁶The dependence of the definition of $\mathbb{O}_{\mathcal{C}}(\mathcal{D})$ on F is implicit in the notation.

above, and denote its pullback along the first element map $h_{\mathcal{D}} : \mathbb{D} \rightarrow \mathcal{D}$ (see Lemma 3.3) by $\mathbb{O}(F^*\mathcal{D}) \rightarrow \mathbb{D}$. Finally, take the power fibration

$$(\mathbb{F}^*\mathbf{E})^{\mathbb{O}(F^*\mathcal{D})} \rightarrow \mathbb{D}.$$

The Δ -structure, as usual, gives us the lax realisation morphism

$$\begin{array}{ccc} (\mathbb{F}^*\mathbf{E})^{\mathbb{O}(F^*\mathcal{D})} & \xrightarrow{|-|} & \mathbb{F}^*\mathbf{E} \\ & \searrow & \swarrow \\ & \mathbb{D} & \end{array}$$

defined by taking $X \in (\mathbb{F}^*\mathbf{E})^{\mathbb{O}(F^*\mathcal{D})}$, which is a functor $\mathbb{D}(F(d_0)) \rightarrow \mathbf{E}(\mathbf{d}_{[n]})$ for some $\mathbf{d}_{[n]} \in \mathbb{D}$, and realising it (cf. Definition 3.8). There is also, however, the 'evaluation' map

$$\begin{array}{ccc} (\mathbb{F}^*\mathbf{E})^{\mathbb{O}(F^*\mathcal{D})} & \xrightarrow{ev} & \mathbb{F}^*\mathbf{E} \\ & \searrow & \swarrow \\ & \mathbb{D} & \end{array}$$

given by sending the same X to $X(d_0)$, since $d_0 \in \mathbb{D}(F(d_0))$. The inclusion $X(d_0) \rightarrow \coprod_{d \in \mathbb{D}(F(d_0))} X(d)$ defines a natural transformation $i : ev \Rightarrow |-|$.

Definition 5.15. Given a resolution $F : \mathcal{D} \rightarrow \mathcal{C}$, a homotopical Δ -opfibration $\mathcal{E} \rightarrow \mathcal{C}$ is *F-special* iff for each $X \in (\mathbb{F}^*\mathbf{E})^{\mathbb{O}(F^*\mathcal{D})}$, which, when viewed as a functor $\mathbb{D}(F(d_0)) \rightarrow \mathbf{E}(\mathbf{d}_{[n]})$, sends all maps of $\mathbb{D}(F(d_0))$ to weak equivalences in $\mathbf{E}(\mathbf{d})$, the X -th component of the natural transformation i ,

$$i_X : ev(X) \rightarrow |X|.$$

is a weak equivalence

The result of this section is the following. Let $F : \mathcal{D} \rightarrow \mathcal{C}$ be a resolution.

Proposition 5.16. *For a F-special (Definition 5.15) homotopical Δ -opfibration $\mathcal{E} \rightarrow \mathcal{C}$ and a locally constant $B \in \mathbb{R}\mathbf{Sect}(\mathcal{D}, \mathcal{E})$, the map*

$$id_{\mathbb{D}!} id_{\mathbb{D}}^* B \rightarrow \mathbb{F}^* \mathbb{F}_! B$$

is a weak equivalence.

We will prove that for each \mathbf{d} , the map $id_{\mathbb{D}!} id_{\mathbb{D}}^* B(\mathbf{d}) \rightarrow \mathbb{F}^* \mathbb{F}_! B(\mathbf{d})$ is an equivalence.

Definition 5.17. Given a $(F : \mathcal{D} \rightarrow \mathcal{C}, c)$ -structure (I, \mathcal{J}, R) and a $(F' : \mathcal{D} \rightarrow \mathcal{C}', c')$ -structure (I', \mathcal{J}', R') , a *morphism* from the first to the second one consists of

- a functor $G : \mathcal{C} \rightarrow \mathcal{C}'$ in $\mathcal{D} \backslash \mathbf{Cat}$ with $G(c) = c'$.

- a commutative square in \mathbf{Cat}/\mathcal{D}

$$\begin{array}{ccc} I & \xleftarrow{R} & J \\ \lambda \downarrow & & \downarrow \mu \\ I' & \xleftarrow{R'} & J'. \end{array}$$

Example 5.18. In Example 5.8, there is a morphism from the second to the first example as soon as $c = F(d)$. In detail: we have a $(F, F(d))$ transition structure $L : \mathcal{D}/d \rightleftharpoons \mathcal{D}(F(d))/d : R$ and $(F, c = F(d))$ transition structure $L' : F/c \rightleftharpoons \mathcal{D}(c) : R'$. In the notation of the definition, G is simply given by $id_{\mathcal{C}}$ (this works because $F(d) = c$), λ is given by mapping $\alpha : d' \rightarrow d$ to $(d', F(\alpha) : F(d') \rightarrow c)$ and μ is the evident functor $\mathcal{D}(c)/d \rightarrow \mathcal{D}(c)$. In this case, even more is true: the square with left adjoints

$$\begin{array}{ccc} \mathcal{D}/d & \xrightarrow{L} & \mathcal{D}(c)/d \\ \lambda \downarrow & & \downarrow \mu \\ F/c & \xrightarrow{L'} & \mathcal{D}(c). \end{array}$$

commutes up to isomorphism.

Remark 5.19. Given functors $p : A \rightarrow B$, $q : A' \rightarrow B$ and $r : A' \rightarrow A$ such that $pr = q$, for any other functor $X : A \rightarrow \mathcal{M}$ to a cocomplete category, there is a natural map $q_! r^* X \rightarrow p_! X$ where as usual, r^* denotes pullback and $p_!, q_!$ denote pushforward functors (left adjoint to pullbacks p^* and q^*).

Lemma 5.20. Fix $\mathbf{c} \in \mathcal{C}$ and $\mathbf{c}' \in \mathcal{C}'$. Let (I, J, R) be a $(F : \mathcal{D} \rightarrow \mathcal{C}, c_0)$ -structure and (I', J', R') be a $(F' : \mathcal{D} \rightarrow \mathcal{C}', c'_0)$ -structure. For any morphism $(G : \mathcal{C}' \rightarrow \mathcal{C}, \lambda, \mu)$ of these transition structures such that $\mathbb{G}(\mathbf{c}') = \mathbf{c}$, a homotopical Δ -opfibration $\mathcal{E} \rightarrow \mathcal{C}$ and a presection $B : \mathbb{D} \rightarrow \mathbb{F}^* \mathbf{E}$, there is an induced morphism of spans

$$\begin{array}{ccccc} \pi_1^* \Pi(\mathbf{c}'_! R_{c'_0} I'^* B) & \longleftarrow & \Pi \mathbf{c}'_! pr_{\mathbb{I}'/\mathbb{R}'}^* B & \longrightarrow & \pi_2^* \Pi(\mathbf{c}'_! J'^* B) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_1^* \Pi(\mathbf{c}_! R_{c_0} I^* B) & \longleftarrow & \Pi \mathbf{c}_! pr_{\mathbb{I}/\mathbb{R}}^* B & \longrightarrow & \pi_2^* \Pi(\mathbf{c}_! J^* B). \end{array} \tag{5.8}$$

Proof. The maps exist due to Remark 5.19. To get the rightmost map of (5.8), apply π_2^* to

$$\Pi(\mathbf{c}'_! J'^* B) \rightarrow \Pi(\mathbf{c}_! J^* B)$$

which we get due to the fact that $\mu^* \mathbf{c}_! J^* B = \mathbf{c}'_! J'^* B$. The middle map of (5.8) is obtained in this way as well, and so is the leftmost map (observe that due to the conditions imposed, both restriction functors R_{c_0} and $R_{c'_0}$ agree).

One can then check the commutativity of the squares obtained through a direct computation. For example, observe that the middle map, in components

$$\coprod_{\substack{\mathbf{i}'_{[m]}, \mathbf{j}'_{[l]}, \\ \alpha': i'_m \rightarrow Rj'_0}} \mathbf{c}'_! B(I'(\mathbf{i}'_{[m]}) *^{I'\alpha'} j'(\mathbf{j}'_{[l]})) \rightarrow \coprod_{\substack{\mathbf{i}_{[m]}, \mathbf{j}_{[l]}, \\ \alpha: i_m \rightarrow Rj_0}} \mathbf{c}_! B(I(\mathbf{i}_{[m]}) *^{I\alpha} j(\mathbf{j}_{[l]}))$$

is induced by the maps of sets indexing the coproducts, given by $(\mathbf{i}', \mathbf{j}', \alpha') \mapsto (\lambda(\mathbf{i}'), \mu(\mathbf{j}'), \lambda\alpha')$ (cf. the proof of Lemma 4.8). \square

Corollary 5.21. *Given a map of two transition structures and a derived section B , the following are equivalent*

1. $|\Pi(\mathbf{c}'_! R_{c'_0} I'^* B)| \rightarrow |\Pi(\mathbf{c}_! R_{c_0} I^* B)|$ is a weak equivalence,
2. $|\Pi(\mathbf{c}'_! j'^* B)| \rightarrow |\Pi(\mathbf{c}_! j^* B)|$ is a weak equivalence.

Proof. Evident. \square

We now apply that to Example 5.18. Observe that for $\mathbf{d} \in \mathbb{D}$ with $\mathbf{c} = \mathbb{F}(\mathbf{d})$, the map

$$id_{\mathbb{D}!} id_{\mathbb{D}}^* B(\mathbf{d}) \rightarrow \mathbb{F}^* \mathbb{F}_! B(\mathbf{d})$$

exactly corresponds to the first morphism in Corollary 5.21. Writing \mathbf{d} instead of \mathbf{c}' , observe that the objects in the second map of Corollary 5.21 are

$$\begin{aligned} \Pi(\mathbf{d}_! j'^* B)_m &= \coprod_{\mathbf{d}'_m \in \mathbb{D}(F(d_0)), d'_m \rightarrow d_0} \mathbb{F}(\mathbf{d})_! B(\mathbf{d}'_{[m]}), \\ \Pi(\mathbf{c}_! j^* B)_m &= \coprod_{\mathbf{d}'_m \in \mathbb{D}(F(d_0))} \mathbb{F}(\mathbf{d})_! B(\mathbf{d}'_{[m]}). \end{aligned}$$

The realisation of the first object is equivalent to $\mathbb{F}(\mathbf{d})_! B(d_0)$. It is easy to check that for a F -special homotopical Δ -opfibration the functor

$$\mathbb{D}(F(d_0)) \rightarrow \mathbf{E}(\mathbb{F}(\mathbf{d})), \quad \mathbf{d}' \mapsto \mathbb{F}(\mathbf{d})_! B(\mathbf{d}')$$

which sends all morphisms to weak equivalences has its realisation equivalent to $\mathbb{F}(\mathbf{d})_! B(d_0)$ and this implies that the map $|\Pi(\mathbf{d}_! j'^* B)| \rightarrow |\Pi(\mathbf{c}_! j^* B)|$ is an equivalence.

Lemma 5.22. *For $F : \mathcal{D} \rightarrow \mathcal{C}$ a resolution, the functor \mathbb{F}^* reflects the condition of being a derived section. That is, if $\mathbb{F}^* A$ is a derived section, then A is one as well.*

Proof. If $\mathbb{F}^* A$ is a derived section for $A \in \mathbf{PSect}(\mathcal{C}, \mathcal{E})$, then take any anchor map $\mathbf{c}' \rightarrow \mathbf{c}$ and find an anchor map $\mathbf{d}' \rightarrow \mathbf{d}$ such that $\mathbb{F}(\mathbf{d}' \rightarrow \mathbf{d}) = \mathbf{c}' \rightarrow \mathbf{c}$ (this is possible due to F being an opfibration with contractible, and hence nonempty, fibers). Then since $\mathbb{F}^* A(\mathbf{d}' \rightarrow \mathbf{d}) = A(\mathbf{c}' \rightarrow \mathbf{c})$, we get that A is a derived section. \square

Corollary 5.23 (proof of Theorem 5.5). $\mathbb{F}_!$ sends locally constant sections to derived sections, and

$$\mathbb{F}_! : \mathrm{Ho} \mathbb{R}\mathrm{Sect}_{lc}(\mathcal{D}, \mathcal{E}) \rightleftarrows \mathrm{Ho} \mathbb{R}\mathrm{Sect}(\mathcal{C}, \mathcal{E}) : \mathbb{F}^*$$

is an equivalence of categories for a special homotopical Δ -fibration $\mathcal{E} \rightarrow \mathcal{C}$.

Proof. We proved that the unit correspondence gives an isomorphism $id \rightarrow \mathbb{F}^*\mathbb{F}_!$ of functors on $\mathrm{Ho} \mathbb{P}\mathrm{Sect}_{lc}(\mathcal{D}, \mathcal{E})$. Using Lemma 5.22, we see that then $\mathbb{F}_!$ preserves derived section condition for locally constant sections. This allows us to restrict the unit $id \rightarrow \mathbb{F}^*\mathbb{F}_!$ to the derived sections. \square

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